The Binomial Model

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ONE-PERIOD MODELS

- Consider two dates: $t = 0$ (today) and $t = 1$.
- Suppose there are only two assets, a bond and stock.
- At time t the price of the bond is B_t and that of the stock is S_t.
- We have two price processes: $B = \{B_0, B_1\}$ and $S = \{S_0, S_1\}$.
- Let r be the interest rate that applies over the period $[0, 1]$. If $B_0 = 1$ then $B_1 = 1 + r$.
- Let $S_0 = s$ and $S_1 = Zs$ where $Z = u$ with probability p_u and $Z = d$ with probability p_d.

Value Process of a Portfolio

If we a portfolio $h = (x, y)$ consisting x bonds and y units of the stock the the value process of the portfolio is given by

$$V_t^h = xB_t + yS_t.$$

That is

$$\begin{aligned} V_0^h &= x + ys \\ V_1^h &= x(1 + r) + ysZ \end{aligned}$$

where $Z = u$ or $Z = d$.

Arbitrage Portfolio

Definition

An arbitrage portfolio is a portfolio h with the properties

$$\begin{aligned} V_0^h &= 0 \\ V_1^h &> 0 \text{ with probability } 1 \end{aligned}$$

A necessary and sufficient condition for absence of arbitrage.

Proposition

The binomial model is free of arbitrage if and only if

$$d \leq 1 + r \leq u.$$

(*Proof on the board*)

Risk-Neutral Probabilities

In the absence of arbitrage, we can find weights q_u and q_d with $q_u, q_d \geq 0$ and $q_u + q_d = 1$ such that

$$1 + r = uq_u + dq_d.$$

Here we may define a new probability measure Q such that $Q(u) = q_u$ and $Q(d) = q_d$ where it can be shown that

$$\begin{aligned} q_u &= \frac{(1+r)-d}{u-d} \\ q_d &= \frac{u-(1+r)}{u-d}. \end{aligned}$$

Risk-Neutral Probabilities vs. Objective Probabilities

- The given probabilities $P(u) = p_u$ and $P(d) = p_d$ are called objective probabilities. They tell us which events are possible and which are not.
- The probabilities q_u and q_d are called risk-neutral probabilities.
- Under this risk-neutral probability measure, we have

$$E^Q(Z) = 1 + r$$

where $E^Q(\cdot)$ denotes expectation under the risk-neutral probability measure.

Definition

A probability measure Q is called a *martingale measure* if the following holds:

$$S_0 = \frac{1}{1+r} E^Q(S_1).$$

Note: Under the measure Q, today's stock price is the discounted expected value of the future stock price.

Arbitrage-free Model

Proposition

The market model is arbitrage-free if and only if there exists a martingale measure Q.

Contingent Claims (Financial Derivatives)

Definition

A *contingent claim(financial derivative)* is a stochastic variable X of the form $X = \Phi(Z)$ where Z is the stochastic variable driving the stock price process.

A European Call under the Binomial Model

Consider a European call option with $sd < K < su$. Then

$$\Phi(Z) = \begin{cases} su - K, & \text{if } Z = u \\ 0, & \text{if } Z = d \end{cases}$$

Reachable Claims and Complete Markets

Definition

A given contingent is said to be reachable if there exists a portfolio h such that

$$V_1^h = X$$

with probability 1. In that case, we say that the portfolio h is a *hedging portfolio* or a *replicating portfolio*. If all claims can be replicated, we say that the market is *complete*.

Pricing a Financial Derivative

Here, the price (value) of a contingent claim X at time t is denoted by $\Pi(t; X)$.

Proposition

Suppose that a claim X is reachable with replicating portfolio h. Then the arbitrage-free price of the claim is

$$\Pi(0; X) = V_0^h.$$

The One-Period Binomial Model is Complete

We can actually show that there exists a portfolio $h = (x, y)$ such that

$$V_1^h = X.$$

Indeed, we have

$$\begin{aligned} x &= \frac{1}{1+r} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d} \\ y &= \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d} \end{aligned}$$

Proposition

Then the arbitrage-free price of a contingent claim X is given by

$$\Pi(0; X) = \frac{1}{1+r} E^Q(X)$$

where Q is the unique martingale measure that satisfies

$$S_0 = \frac{1}{1+r} E^Q(S_1).$$

Example 1

A stock price is currently \$40.It is known that at the end of one month, it will be either \$42 or \$48.The risk-free rate 8% per annum with monthly compounding. What is the value of a one-month European call option with a strike of \$39.

Example 2: Two-period model

A stock price is currently \$100. Over each of the next two 6-month periods, it is expected to go up by 10% and down by 10%. The risk-free rate is 8% per annum with semi-annual compounding. What is the value of a one-year European call option with a strike of \$100.

Multi-period Models

- Multi-period models are actually used in practice. The more time steps we use the better is the approximation of the value of the derivative that we get. (Using the software *Derivagem*)
- As $n \to \infty$, our n-period binomial model becomes a better approximation of another model called Black-Scholes Model.

References

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Arbitrage Theory in Continous Time, 2nd ed.

The End