THIRD LEE PENG YEE SYMPOSIUM
Celebrating Mathematics
National Institute of Education
Nanyang Technological University
18-19 November 2013

THE HENSTOCK APPROACH TO THE ITÔ STOCHASTIC INTEGRAL

EMMANUEL A. CABRAL, PH.D.
ATENEO DE MANILA UNIVERSITY, PHILIPPINES

THE HENSTOCK APPROACH

The function

$$F(x) = \begin{cases} x^2 \sin\frac{1}{x^2}, & x > 0 \\ 0, & x = 0 \end{cases}$$

is differentiable everywhere with

$$f(x) = F'(x) = \begin{cases} 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}, & x > 0 \\ 0, & x = 0 \end{cases}$$

However, $\int_0^x F'(t)dt \neq F(x)$ in the sense of Riemann or Lebesgue integration.

THE HENSTOCK APPROACH

Definition A function f is Henstock integrable on $[a,b]$ if for some number A, the following holds: for every $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that

$$\left| A - \sum_{i=1}^{n} f(t_i)(x_{i+1} - x_i) \right| < \varepsilon$$

whenever

$$D = \{(t_i, [x_i, x_{i+1}])\}_{i=1}^{n}$$

is a division[1] of $[a,b]$ with

$$t_i \in [x_i, x_{i+1}] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$$

for every point-interval pair in the division D.

We write

$$\int_a^b f(x)dx = A\,.$$

[1] A division of $[a,b]$ consists of point-interval pairs such that the union of the intervals is $[a,b]$. The division being described above is called a δ −fine division.

THE HENSTOCK APPROACH

If a function f is Henstock integrable on $[a, b]$, then it is also Henstock integrable in every subinterval $[c, d]$ of $[a, b]$. It is then natural to define the *primitive* F *as*

$$F(x) = \int_a^x f(x)dx$$

which can also be taken as a function of intervals in the sense that

$$F(u, v) = \int_a^v f(x)dx - \int_a^u f(x)dx.$$

THE HENSTOCK APPROACH

Cousin's Lemma For every positive function δ on $[a, b]$, there exists a δ −fine division of $[a, b]$.

Henstock's Lemma Suppose a function f is Henstock integrable on $[a, b]$. Then for every $\varepsilon > 0$ there exists a function δ on $[a, b]$ such that

$$\sum_{i=1}^{n} \left| \big(F(x_{i+1}) - F(x_i)\big) - f(t_i)(x_{i+1} - x_i) \right|$$

whenever $D = \{(t_i, [x_i, x_{i+1}])\}_{i=1}^{n}$ is a δ −fine *partial division* of $[a, b]$.

THE HENSTOCK APPROACH

Decomposability property

Suppose $f_n \to f$ and each f_n is Henstock integrable with primitive F_n. Then δ_n could be found for each f_n that satisfies the integrability condition.

One may subdivide $[a, b]$ into a countable union of pairwise disjoint sets X_n such that $\delta(x) = \delta_n(x)$ for $x \in X_n$. This is the δ that we need to prove integrability of f given other conditions on f_n and F_n.

FUNDAMENTAL THEOREM OF CALCULUS

A function f is Henstock integrable with primitive F *if and only if* F is ACG* and $F'(x) = f(x)$ almost everywhere.

A function f is Lebesgue integrable with primitive F *if and only if* F is AC and $F'(x) = f(x)$ almost everywhere.

HENSTOCK APPROACH TO THE LEBESGUE INTEGRAL

A function f is McShane integrable on $[a,b]$ if for every $\varepsilon > 0$ there exists a positive function $\delta > 0$ such that

$$\left| A - \sum_{i=1}^{n} f(t_i)(x_{i+1} - x_i) \right|$$

whenever

$$D = \{(t_i, [x_i, x_{i+1}])\}_{i=1}^{n}$$

is a δ –fine McShane division of $[a,b]$.

RUDIMENTS OF PROBABILITY THEORY

- Sample space Ω
- Events
- σ −algebra or σ −field $\mathcal{F}$
- Filtration $\{\mathcal{F}_t\}$
- Probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$
- Probability measure
- Random variable
- Expectation
- Conditional expectation

STOCHASTIC PROCESSES

- A stochastic process $f: [0, T] \times \Omega \to \mathbb{R}$ is a collection of random variables $f_t, t \in [0, T]$. We may write $f = \{f_t\}$.
- A process f is adapted to the filtration $\{\mathcal{F}_t\}$ if for every $t \in [0, T]$, f_t is $\mathcal{F}_t$ −measurable.

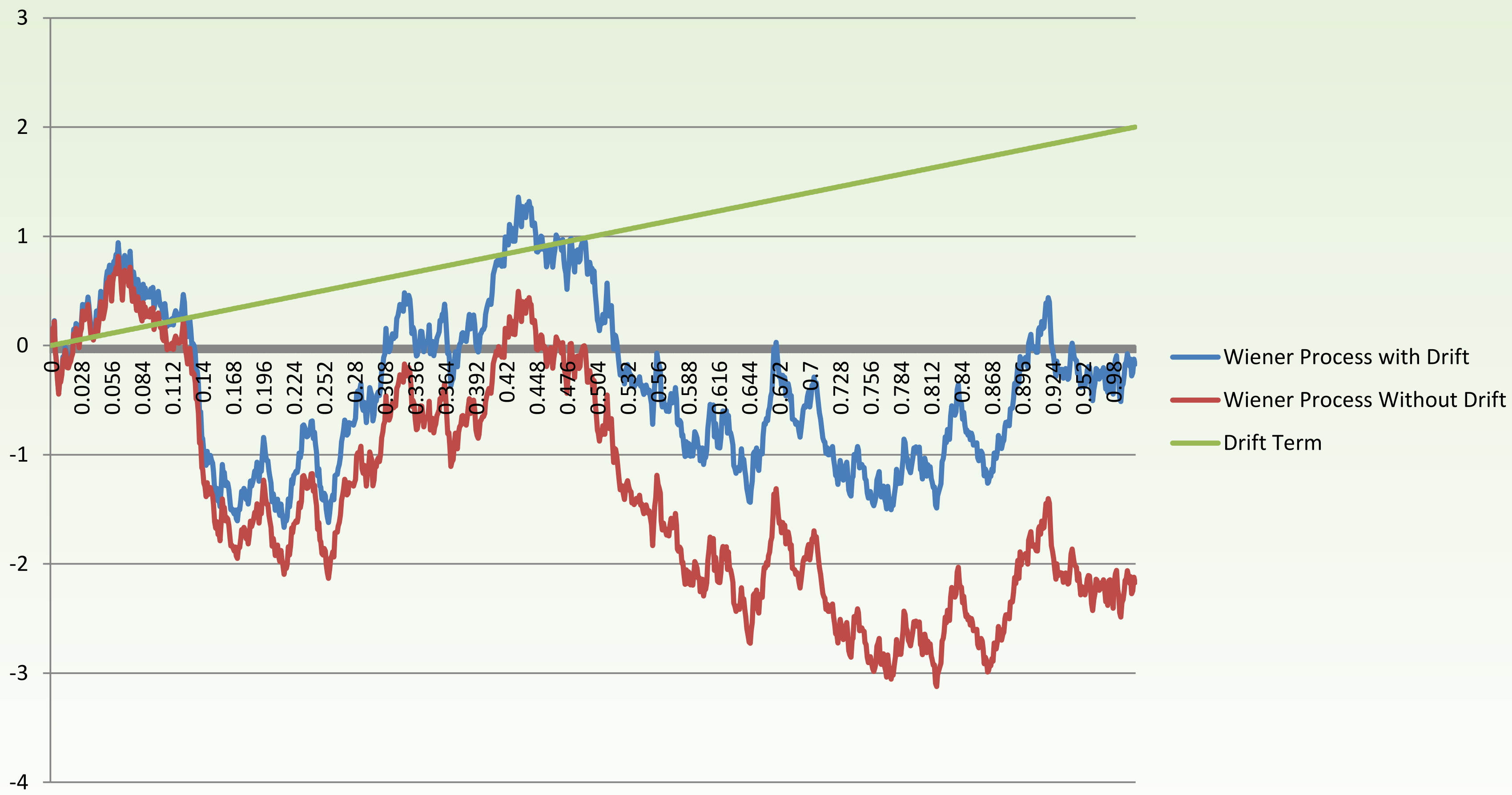
WIENER PROCESS

- Also known as Brownian motion.
- A process $\{W_t\}$ is a Wiener process if
  - $W_{t+\Delta t} - W_t$ is normally distributed for each t.
  - Mean of $W_{t+\Delta t} - W_t$ is 0.
  - Variance of $W_{t+\Delta t} - W_t$ is Δt.
  - For $0 \leq u < v \leq s < t$, $W_v - W_u$ and $W_t - W_s$ are independent random variables.
- Standard filtration is given by $\mathcal{F}_t = \sigma(W_s : 0 \leq s \leq t)$.

BROWNIAN MOTION WITH DRIFT

- If a and b are constants, then we have a Brownian motion with drift $adt + bdW_t$, also written $\int_0^t ads + \int_0^t bdW_s$.
- In general a and b could be functions of t and W_t and then we have an Itô process.
- Appears very often in financial mathematics models.

WIENER PROCESS WITH CONSTANT DRIFT



CONSTRUCTION OF THE CLASSICAL ITÔ INTEGRAL: SIMPLE PROCESSES

Definition *A stochastic process* $f = \{f_t\}_{0 \le t \le T}$ *is called a simple process if it can be written as*

$$f_t(\omega) = \sum_{i=1}^{p} \phi_i(\omega)\, \mathbf{1}_{(t_{i-1},\, t_i]}(t)$$

where

$$0 = t_0 < t_1 < \cdots < t_p = T$$

and ϕ_i *is* $\mathcal{F}_{t_{i-1}}$ *−measurable and bounded.*

CONSTRUCTION OF THE CLASSICAL ITÔ INTEGRAL: STOCHASTIC INTEGRAL OF A SIMPLE PROCESS

Definition *For* $t \in (t_k, t_{k+1}]$, *the stochastic integral* $I(f)_t$ *over* $[0, t]$ *of a simple process is defined by*

$$I(f)_t = \sum_{i=1}^{k} \phi_i \left(W_{t_i} - W_{t_{i-1}} \right) + \phi_{k+1} \left(W_t - W_{t_k} \right),$$

which can also be compactly written as

$$I(f)_t = \sum_{i=1}^{p} \phi_i \left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t} \right).$$

We also write

$$\int_0^t f_s dW_s = \sum_{i=1}^{p} \phi_i \left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t} \right).$$

CONSTRUCTION OF THE CLASSICAL ITÔ INTEGRAL

How then do we give meaning to

$$\int_0^T f_t dW_t$$

if f is not a simple process?

This integral cannot be defined pathwise, that is for every $\omega \in \Omega$ similar to the Lebesgue-Stieltjes integral precisely because the Wiener sample paths are of unbounded variation.

CONSTRUCTION OF THE CLASSICAL ITÔ INTEGRAL

Definition 3.4 Let $\mathcal{L}_2$ be the space of stochastic process $f : [0, T] \times \Omega \to \mathbb{R}$, such that

(i) f is $\mathcal{B} \times \mathcal{F}_t$ −measurable where $\mathcal{B}$ denotes the Borel σ-algebra on $[0, t]$;

(ii) the process $\{f_t(\omega)\}_{t \in [a,b]}$ is adapted to the filtration $\mathcal{F}_t$; and

(iii) $\|f\|_{\mathcal{L}_2}^2 = \mathbb{E}\left((L) \int_a^b f_t^2 dt\right) < \infty$.

CONSTRUCTION OF THE CLASSICAL ITÔ INTEGRAL

Theorem *For every* $f \in \mathcal{L}_2$*, there exist a sequence of simple processes* $f^n = \{f_t^n\}$ *and a unique random variable* $F_T \in L^2$ *such that*

$$\mathbb{E}\left[\int_0^T |f_s^n - f_s|^2 ds\right] \to 0$$

and

$$\mathbb{E}[(I(f^n)_T - F_T)^2] \to 0.$$

We call F_T *the stochastic integral of* f *over the interval* $[0, T]$ *and we write*

$$\int_0^T f_s dW_s = F_T.$$

CONSTRUCTION OF THE CLASSICAL ITÔ INTEGRAL

It can be shown that

$$\int_0^t f_s dW_s$$

also exists for $0 \leq t \leq T$. It is then natural to define the process $F = \{F_t\}_{0 \leq t \leq T}$ where

$$F_t = \int_0^t f_s dW_s \, .$$

CONSTRUCTION OF THE CLASSICAL ITÔ INTEGRAL

Theorem *Suppose* $f \in \mathcal{L}_2$. *Then the process* $F = \{F_t\}_{0 \le t \le T}$ *defined by*

$$F_t = \int_0^t f_s dW_s$$

has the following properties

1) *Ito Isometry, that is*

$$\mathbb{E}\left[\left(\int_0^t f_s dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t f_s^2 ds\right].$$

2) *Martingale property, that is* F *is adapted to the filtration* $\mathcal{F} = \{\mathcal{F}_t\}$, $\mathbb{E}(f_t) < \infty$, *and for all* s, t with $0 \le s \le t \le T$, we have

$$\mathbb{E}[f_t | \mathcal{F}_s] = f_s.$$

THE HENSTOCK APPROACH TO THE ITÔ STOCHASTIC INTEGRAL

Definition *Let* $f = \{f_s : s \in [0,T]\}$ *a stochastic process adapted to the standard filtering space* $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, \mathbf{P})$*. Then* f *is said to be Itô integrable on* $[0,T]$ *if there exists a random variable* $F_T \in L^2$ *such that for* $\varepsilon > 0$*, there exists a positive function* δ *on* $[0,T)$ *and a positive number* η *such that*

$$\mathbb{E}\left[\left(F_T - \sum_{i=1}^{n} f_{u_i}(W_{v_i} - W_{u_i})\right)^2\right] < \varepsilon$$

whenever $D = \{([u_i, v_i), u_i)\}_{i=1}^{n}$ *is a* δ*-fine belated partial division of* $[0,T)$ *with*

$$\left|T - \sum_{i=1}^{n}(v_i - u_i)\right| < \eta.$$

Consider L^2 to be the space of random variables X such that $\|X\|_{L^2} = \mathbb{E}(|X|^2) < \infty$.

THE HENSTOCK APPROACH TO THE ITÔ STOCHASTIC INTEGRAL

Henstock Lemma *Let f be Itô integrable on $[0,T]$ with*

$$F_t = \int_0^t f_s dW_s \,.$$

Then for every $\varepsilon > 0$ there exists a positive function $\delta > 0$ on $[0,T)$ such that

$$\sum_{i=1}^{n} \mathbb{E}\left|\left(F_{v_i} - F_{u_i}\right) - f_{u_i}\left(W_{v_i} - W_{u_i}\right)\right|^2 < \varepsilon$$

whenever $D = \{(\,[u_i, v_i), u_i)\}_{i=1}^{n}$ is δ –fine partial division of $[0,T)$.

THE HENSTOCK APPROACH TO THE ITÔ STOCHASTIC INTEGRAL

Definition *The stochastic process* $X = \{X_t\}$ *is said to have the* AC^2 *property if for every* $\varepsilon > 0$*, there exists a positive number* η *such that*

$$\mathbb{E}\left[\left(\sum_{i=1}^{n}\left(F_{v_i} - F_{u_i}\right)\right)^2\right]$$

whenever $\{[u_i, v_i)\}$ *is a finite collection of disjoint subintervals of* $[0, T]$.

AC^2 was introduced by Toh, T.L. and Chew T.S. in the paper *The Non-Uniform Riemann Approach to Itô's Integral,* Real Analysis Exchange 18(1992/1993), 352-366.

THE HENSTOCK APPROACH TO THE ITÔ STOCHASTIC INTEGRAL

Definition *An adapted process* $f = \{f_s : s \in [0,T]\}$ *on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ *is called a* L^2 *−martingale if it satisfies the following:*

1. *For every* $s \in [0,T]$, *we have* $\mathbb{E}(f_s) < \infty$.
2. *For all* s, t *with* $0 \le s \le t \le T$, *we have* $\mathbb{E}[f_t | \mathcal{F}_s] = f_s$.
3. $\sup_{t \in [0,T]} \mathbb{E}(f_t^2) < \infty$.

THE HENSTOCK APPROACH TO THE ITÔ STOCHASTIC INTEGRAL

Definition *The stochastic process* F_t *on* $[0,T]$ *is said to have a* *belated* *derivative* f_t *with respect to the Wiener process if for every* $\varepsilon > 0$, *there exists a positive number* $\delta(t)$ *such that*

$$\mathbb{E}[(F_v - F_t) - f_t(W_v - W_t)]^2 \leq \varepsilon \mathbb{E}(W_v - W_t)^2$$

whenever the pair $([t,v),t)$ *is* δ *−fine. We write*

$$D_b F_t = f_t.$$

Toh, T.L. and Chew T.S., On belated differentiation and a characterization of Henstock-Kurzweil-Ito integrable processes, *Mathematica Bohemica*, Vol. 130 (2005), No. 1, 63--72

THE HENSTOCK APPROACH TO THE ITÔ STOCHASTIC INTEGRAL

Fundamental Theorem of Calculus

The following set of conditions is sufficient and necessary for a stochastic process $f: [0, T] \times \Omega$ to Itô integrable on $[0, T]$:

1. F *is an* L^2 *−martingale;*
2. F *has* AC^2 *property on* $[0, T]$*;*
3. $D_b F_t = f_t$ *for almost every* $t \in [0, T)$.

Toh, T.L. and Chew T.S., On belated differentiation and a characterization of Henstock-Kurzweil-Ito integrable processes, *Mathematica Bohemica*, Vol. 130 (2005), No. 1, 63--72

BACKWARDS STOCHASTIC INTEGRALS[3]

- Filtration $\mathcal{F}_t = \sigma(W_s: 0 \leq s \leq t)$ is replaced by backwards filtration $\mathcal{G}_t = \sigma(W_s: t \leq s \leq T)$.
- Partial divisions are tagged from the right, that is, $D = \{((u_i, v_i], v_i)\}_{i=1}^{n}$.
- $F_{(t,T]} = \int_t^T f_s dW_s$.
- Backwards derivative is defined using backwards filtration.

[3] Backwards Stochastic integral using Henstock approach is given in the paper by Arcede, J.P. and Cabral, E.A.,*An Equivalent Definition for the Backwards Itô Integral,* Thai Journal of Mathematics, 9 (2011), 619 - 630.

BACKWARDS STOCHASTIC INTEGRALS

Example: The Wiener process is backwards integrable with

$$\int_a^b W_s dW_s = \frac{1}{2}\left(W_b^2 - W_a^2\right) + \frac{1}{2}(b-a).$$

BACKWARDS STOCHASTIC INTEGRALS

Example Consider the processes

$$X_{(\xi,T]} = \frac{1}{2}\left(W_T^2 - W_\xi^2\right) + \frac{1}{2}(T - \xi)$$

and

$$\mathrm{Y}_{(\xi,T]} = \frac{1}{2}\left(W_T^2 - W_\xi^2\right).$$

Both have backward derivatives W_ξ. Both processes have the AC^2 property but only the first one is an L^2 −martingale.

One may check out the details in the paper Arcede, J.P. and Cabral, E.A., *Fundamental Theorem of Calculus for the Backward Itô Integral*, Matimyas Matematika, 34(1), 2011, 1-9.