RATIONAL EXPONENTS

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Principal n^{th} Root

- The principal n^{th} root of a positive number b is the **positive** real number that satisfies the equation $x^n - b = 0$.
- This principal root is denoted by $b^{1/n}$ or $\sqrt[n]{b}$
- The n^{th} roots of b are **all** the real numbers that satisfy the above equation.

- If n is odd, the n^{th} root of a real number b is the real number that satisfies the equation

$$x^n - b = 0$$

- If n is even and b is negative, there is no real number that satisfies the equation

$$x^n - b = 0.$$

- We say that $b^{1/n}$ (or $\sqrt[n]{b}$) is not well-defined defined (or does not exist) in the set of real numbers.

- The principal square root of 4 is 2.
- The square roots of 4 are 2 and -2.

Check yourself.

- What is $4^{\frac{1}{2}}$?
- What is $\sqrt{4}$?
- Solve the equation

$$x^2 - 4 = 0$$

- If n is a positive integer and $b^{\frac{1}{n}}$, $a^{\frac{1}{n}}$ both exist as real numbers, the following hold:
- $(ab)^{\frac{1}{n}}$ is a real number and

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$$

- if b is not zero, then $\left(\frac{a}{b}\right)^{\frac{1}{n}}$ is a real number and

$$\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$$

- If m and n are positive integers and $b^{\frac{1}{n}}$ is well-defined in the set of real numbers, then

$$(b^m)^{\frac{1}{n}} = (b^{\frac{1}{n}})^m$$

- However, if $b^{\frac{1}{n}}$ is not a real number then, the above equality is not true. The left-hand side may be well-defined while the right-hand side is not.

Rational Exponent

- Suppose m and n are positive integers, then then expression $b^{\frac{m}{n}}$ is well-defined whenever $b^{\frac{1}{n}}$ is well-defined and

$$b^{\frac{m}{n}} = (b^m)^{\frac{1}{n}} = \left(b^{\frac{1}{n}}\right)^m$$

- It would be meaningless to write $(-64)^{\frac{2}{6}}$.
- Yet, the number $(-64)^{\frac{1}{3}}$ is well-defined.
- $[(-64)^2]^{1/6} = 4$.

- The number $(-2)^{\frac{6}{2}}$ is not well-defined.
- The number $(-2)^3 = -8$.
- The number $[(-2)^6]^{1/2} = 8$.

- So $(-1)^1 = -1$
- However $[(-1)^2]^{1/2} = 1$
- And $[(-1)^{1/2}]^2$ is not well-defined.

A Generalization

- Let m and n be positive integers. Suppose the simplest form of m/n is p/q. If $b^{\frac{m}{n}}, b^{\frac{p}{q}}$ are both well-defined, then the two are equal.
- $(-8)^{2/3} = (-8)^{6/9}$.
- But $(-8)^{2/3}$ is well-defined while $(-8)^{4/6}$ is not and thus the two are not equal.

Absolute Value

- Observe that $[(-2)^4]^6 = (2^4)^6 = 2^{4/6} = 2^{2/3}$.
- If b is negative and m, n are positive even integers then

$$(b^m)^{\frac{1}{n}} = |b|^{\frac{m}{n}} = |b|^{\frac{p}{q}}$$

where p/q is the simplest form of m/n.

Check Yourself

- Simplify $\sqrt{(-2)^2}$
- Simplify $\sqrt{2^2}$
- Simplify $\sqrt{b^2}, b \in \mathbb{R}$