Diliman Mathematics Research Workshop
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THE NELSON-SIEGEL MODEL

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WHY MODEL AND FORECAST INTEREST RATES?

- Valuation of assets and derivative pricing.
- Risk management and regulatory compliance.

MATHEMATICAL THEORY OF INTEREST: SPOT RATES

- A τ –bond (*discount bond*) is a financial security that pays P1 at maturity date τ.
- At any time $t < \tau$, the market price of this τ –bond is $p(t, \tau\ )$. This discount bond is said to have a *remaining maturity* of $\tau - t$ at time t.
- The continuously compounded interest rate $y_t(\tau)$ for the period $[t, \tau]$ implied by this bond price is given by

$$p_t(\tau) \cdot e^{y_t(\tau)\cdot(\tau - t)} = 1.$$

- The rate $y_t(\tau)$ is called the *spot rate* or *zero rate* that applies over the time period $[t, \tau]$.

MATHEMATICAL THEORY OF INTEREST: FORWARD RATES

- Let $t < \tau_1 < \tau_2$. At time t, an investor can sell a τ_1 –bond and use the proceeds to buy $\frac{p_t(\tau_1)}{p_t(\tau_2)}$ units of the τ_2 –bond.
- Later on at time τ_1, the investor pays P1 to the holder of the τ_1 –bond. At time τ_2, the investor receives an amount equal to $\frac{p_t(\tau_1)}{p_t(\tau_2)}$ pesos from the τ_2 –bonds held.
- In effect, the investor contracted at time t to invest P1 at time τ_1 in order to receive a cash flow of $\frac{p(t,\tau_1)}{P(t,\tau_2)}$ pesos at time τ_2.
- This contract implies a continuously compounded *forward rate* $F_t(\tau_1, \tau_2)$ seen at time t that applies for the future period $[\tau_1, \tau_2]$.

MATHEMATICAL THEORY OF INTEREST: FORWARD RATES

- The forward rate $F_t(\tau_1, \tau_2)$ is defined by the equation

$$1 \cdot e^{F_t(\tau_1,\tau_2)\cdot(\tau_2-\tau_2)} = \frac{p_t(\tau_1)}{p_t(\tau_2)}.$$

- That is,

$$F_t(\tau_1, \tau_2) = -\frac{\ln p_t(\tau_2) - \ln p_t(\tau_1)}{\tau_2 - \tau_1}.$$

- The forward rate seen at time t for the future period $[\tau, \tau + \Delta\tau]$ is then

$$F_t(\tau, \tau + \Delta\tau) = -\frac{\ln p_t(\tau + \Delta\tau) - \ln p_t(\tau)}{\Delta\tau}.$$

MATHEMATICAL THEORY OF INTEREST: INSTANTANEOUS FORWARD RATES

- The forward rate seen at time t for the future period $[\tau, \tau + \Delta\tau]$ is then
$$F_t(\tau, \tau + \Delta\tau) = -\frac{\ln p_t(\tau + \Delta\tau) - \ln p_t(\tau)}{\Delta\tau}.$$
- As $\Delta\tau \to 0$, we get the *instantaneous forward rate seen at time t for the future time τ:*
$$f_t(\tau) = -\frac{\partial \ln p_t}{\partial \tau}.$$

MATHEMATICAL THEORY OF INTEREST: SPOT RATES IN TERMS OF INSTANTANEOUS FORWARD RATES

- So

$$\ln p_t(\tau) = -\int_t^\tau f_t(u)du$$

- and

$$y_t(\tau) = -\frac{\ln p_t(\tau)}{\tau - t}$$
$$y_t(\tau) = \frac{\int_t^\tau f_t(u)du}{\tau - t}.$$

- The spot rate is therefore the average over $[t, \tau]$ of the instantaneous forward rates $f_t(u)$.

MATHEMATICAL THEORY OF INTEREST: TERM STRUCTURE

- The term structure of interest rates is described by any one of the following:
  1. *The zero-coupon yield curve* (or simply, *yield curve*) at time t is the graph that expresses the relationship between the spot rate $y_t(\tau)$ and the remaining maturity $\tau - t$ of the corresponding discount bond. For a fixed t, this is the set
  $$\{y_t(\tau) : \tau > t\}.$$
  2. The *discount curve* at time t expresses the relationship between the price $P(t, \tau)$ of a discount bond and its remaining maturity $\tau - t$. This is the set
  $$\{p_t(\tau) : \tau > t\}.$$
  3. The *forward curve* at time t is the relationship between the instantaneous forward rate $f_t(\tau)$ and $\tau - t$, that is the set
  $$\{f_t(\tau) : \tau > t\}.$$

- If any one of these three curves is known at time t then the price of any coupon bond is determined by discounting each future cash flow at the corresponding spot rate.

MATHEMATICAL THEORY OF INTEREST: BASIC GOAL

- The bond market contains an infinite number of types of asset, one asset type for each maturity date τ.
- The basic goal in interest rate theory is to model the bond market, that is, investigate the relations among the different bonds.
- To do this we can
  1. Specify the dynamics of the forward curve.
  2. Specify the dynamics of the discount curve.
  3. Specify the dynamics of the yield curve.

MODEL DYNAMICS[1]

Discount curve dynamics

$$dp_t(\tau) = p_t(\tau)m_t(\tau)dt + p_t(\tau)v_t(\tau)dW_t$$

Forward curve dynamics

$$df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t$$

The above discount curve and forward curve dynamics are both infinite-dimensional systems of stochastic differential equations.

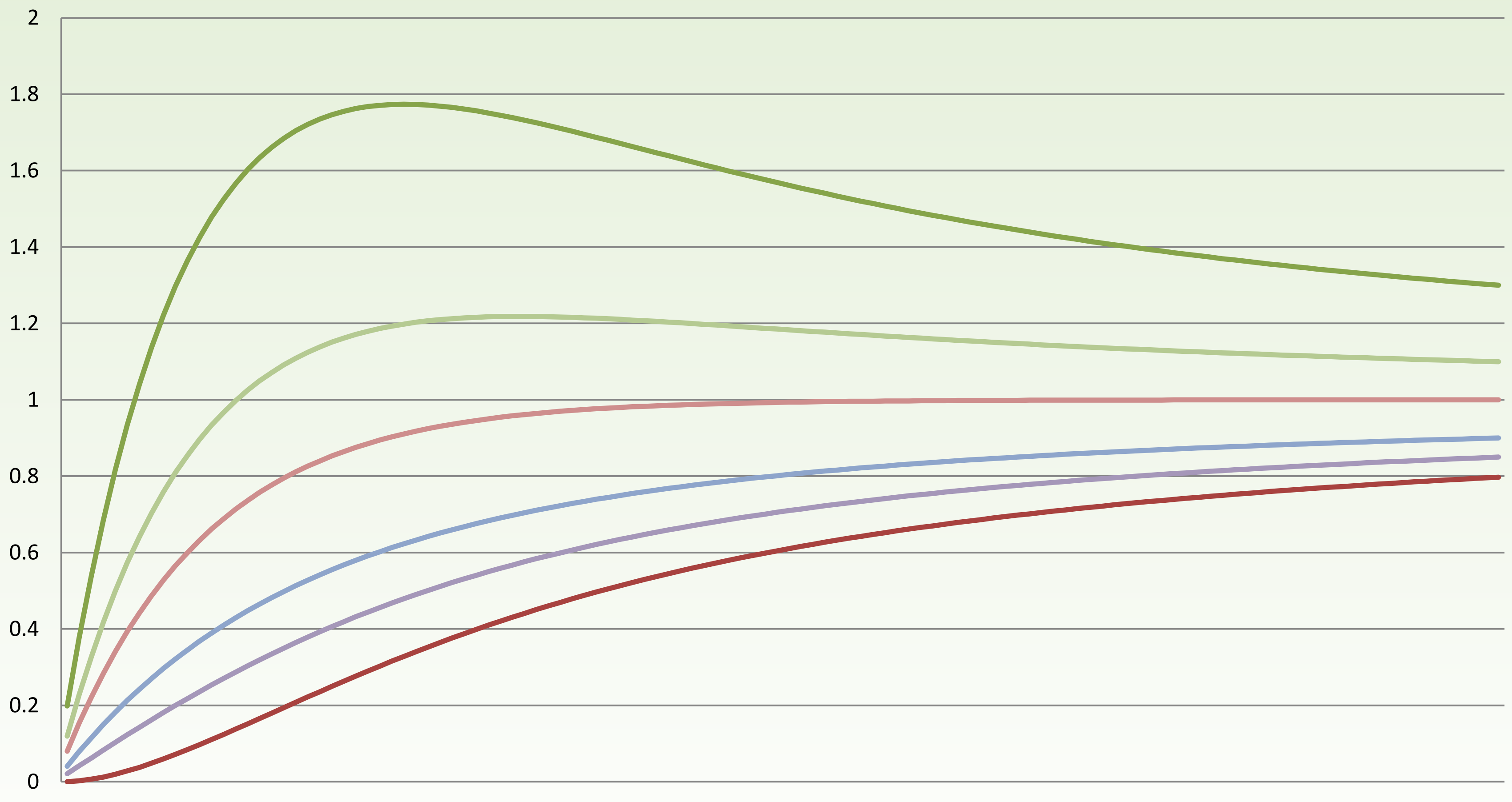
[1]Bjork, T., Arbitrage Theory in Continuous Time

STYLIZED FACTS

- *Stylized facts* are observations that have been made in so many contexts that they are widely understood to be empirical truths, to which theories must fit.[2]
- Long historical records have indicated that estimated yield curves assume a range of shapes that are essentially concave upward sloping, humped, or occasionally S-shaped.[3]

[2]http://economics.about.com/od/economicsglossary/g/stylized.htm
[3]Nelson, C., and Siegel, A., *Parsimonious Modelling of Yield Curves, The Journal of Business,* 60(4), 473-489.

ILLUSTRATION: RANGE OF LONG OBSERVED YIELD CURVE SHAPES



PARSIMONIOUS MODELLING OF YIELD CURVES[4]

- The use of only a few parameters to describe the whole term structure.
- The model must be able to generate the full range of shapes of typically observed yield curves, namely monotonic, humped and S-shaped.
- There is a class of functions that can generate this range of shapes.

[4]Nelson, C., and Siegel, A., *Parsimonious Modelling of Yield Curves, The Journal of Business,* 60(4), 473-489.

THE FORWARD RATE AS A SOLUTION TO A 2ND-ORDER ORDINARY DIFFERENTIAL EQUATION

A second-order differential equation[5] of the form

$$\frac{d^2 f_t(\tau)}{d\tau^2} + \frac{2}{\lambda_t}\frac{d f_t(\tau)}{d\tau} + \frac{1}{\lambda_t^2} f_t(\tau) = \frac{1}{\lambda_t^2}\beta_{0t}$$

has a solution of the form

$$f_t(\tau) = \beta_{0t} + \beta_{1t} e^{-\tau/\lambda_t} + \beta_{2t}\frac{\tau}{\lambda_t} e^{-\tau/\lambda_t}.$$

A little calculus would reveal that[6]

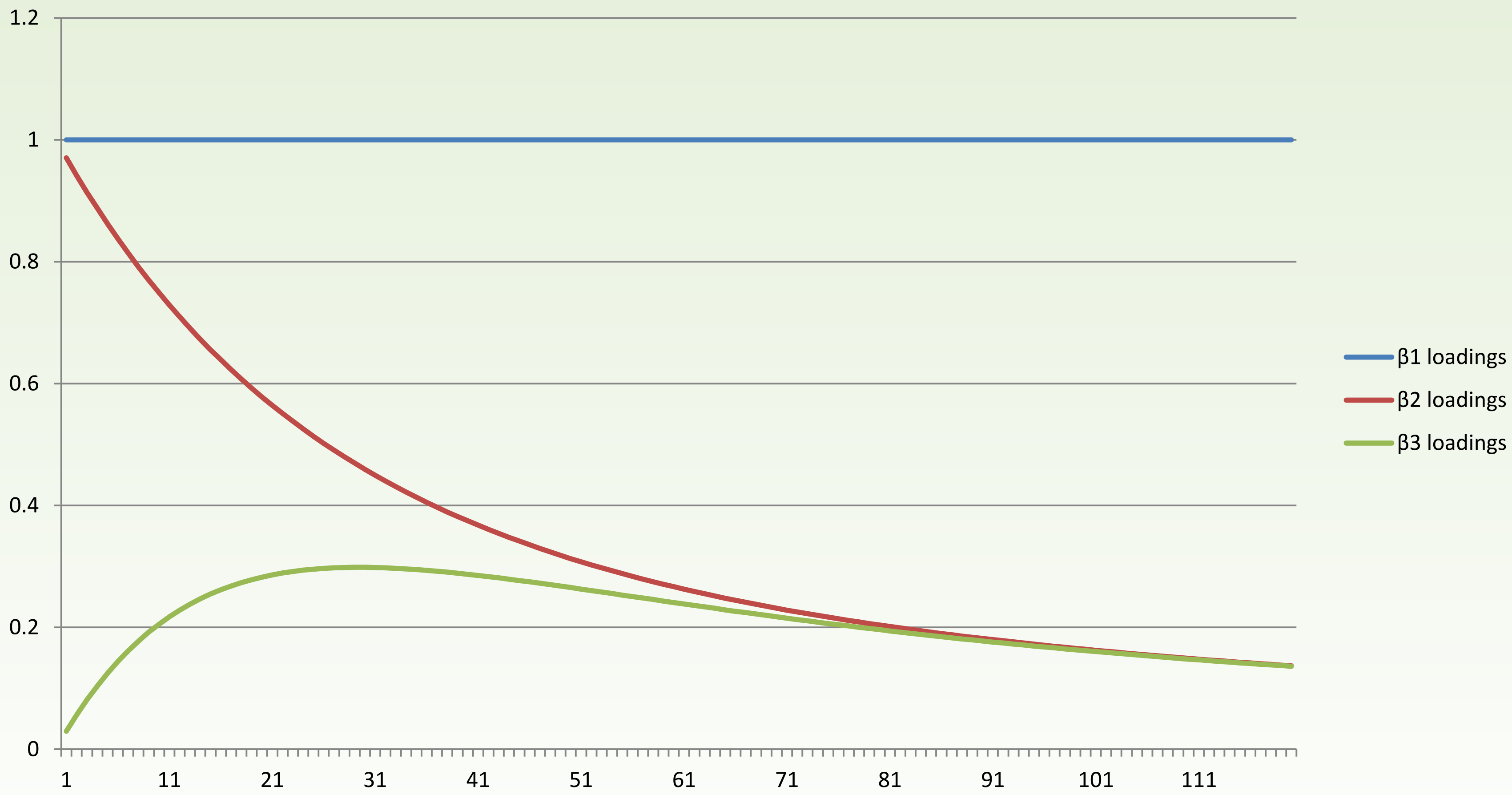
$$y_t(\tau) = \beta_{0t} + \beta_{1t}\left(\frac{1 - e^{-\tau/\lambda_t}}{\tau/\lambda_t}\right) + \beta_{2t}\left(\frac{1 - e^{-\tau/\lambda_t}}{\tau/\lambda_t} - e^{-\tau/\lambda_t}\right)$$

[5]This equation though mentioned in the original Nelson-Siegel paper did not appear explicitly in that paper as shown above. However, familiarity with the theory of ordinary differential equations would allow one to deduce the same equation.
[6]Diebold, F. and Li, C., *Forecasting the Term Structure of Government Bond Yields, Journal of Econometrics,* 130(2006), 337-364.

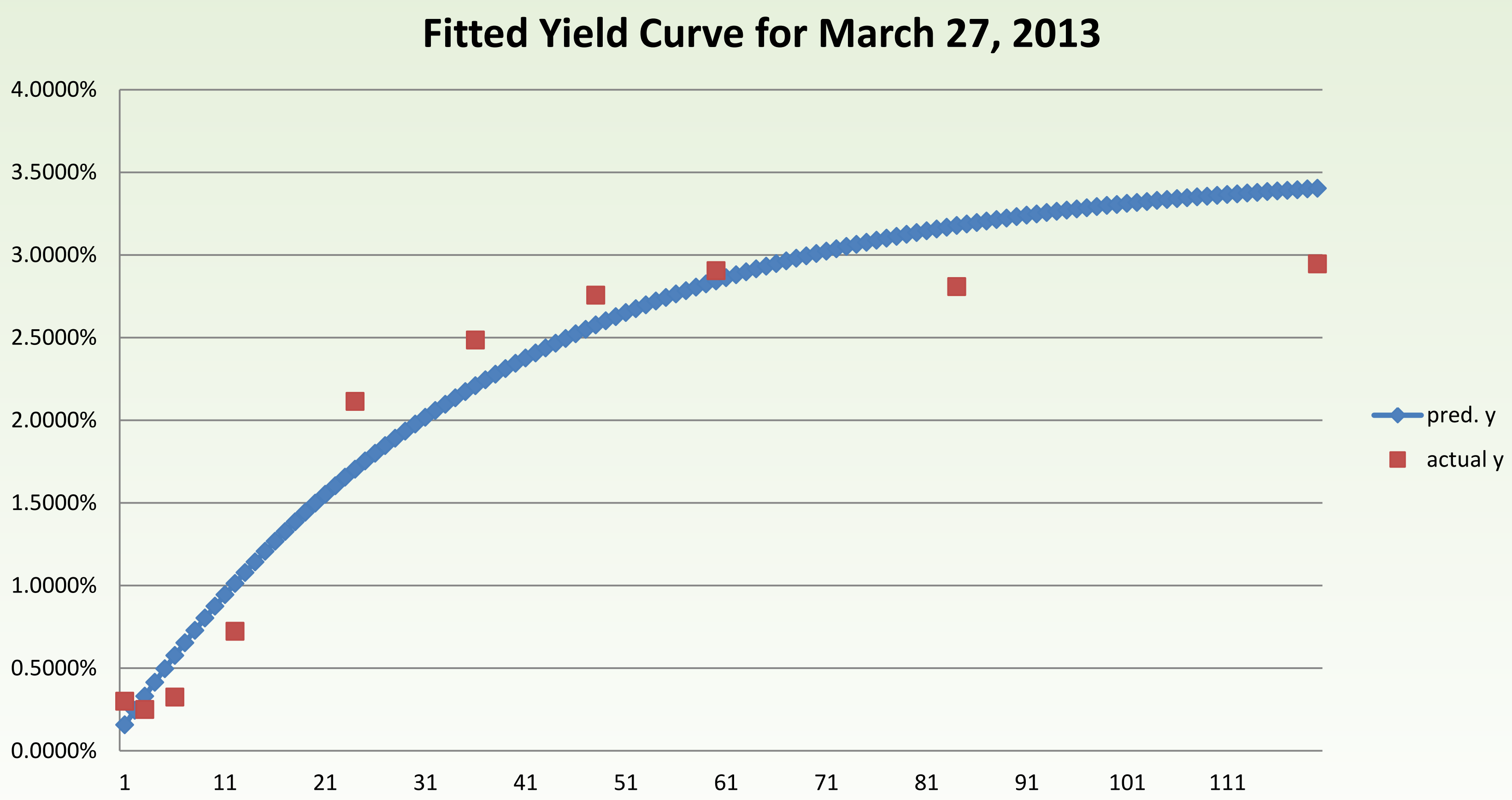
THE PARAMETERS

- The yield curve is a superposition of three *factor loadings* ℓ_1, ℓ_2 *and* ℓ_3:
- $\ell_1 = 1$, $\ell_2 = \frac{1-e^{-\tau/\lambda_t}}{\tau/\lambda_t}$, and $\ell_3 = \frac{1-e^{-\tau/\lambda_t}}{\tau/\lambda_t} - e^{-\tau/\lambda_t}$.
- The strength of each factor loading is determined by the *factors* β_{0t}, β_{1t} and β_{2t} .
- The shape parameter λ_t is chosen to correspond to the value of τ such that ℓ_3 is maximum.
- When λ_t is specified, the problem becomes a multiple linear regression problem.

FACTOR LOADINGS



A FITTED YIELD CURVE



$$\lambda_t = 3.25141$$

THE DATA

- Data consists of 180 sets of yield rates (1M, 3M, 6M, 1Y, 2Y, 3Y, 4Y, 5Y, 7Y, 10Y, 15Y, and 20Y rates) for 180 Wednesdays) from October 7, 2009 to March 27, 2013.
- The first 128 sets will be used to make a 52-week forecast. This is done by forecasting the next set of 52 triples

$$\{(\beta_{0t}, \beta_{1t}, \beta_{2t}): t \in \{1,2,\ldots,52\}\}$$

  using the preceding set of 128 triples

$$\{(\beta_{0t}, \beta_{1t}, \beta_{2t}): t \in \{-127, -126, \ldots, -1, 0\}\}$$

- The last 52 sets of yield rates in the data is to be compared with the 52-week forecast.

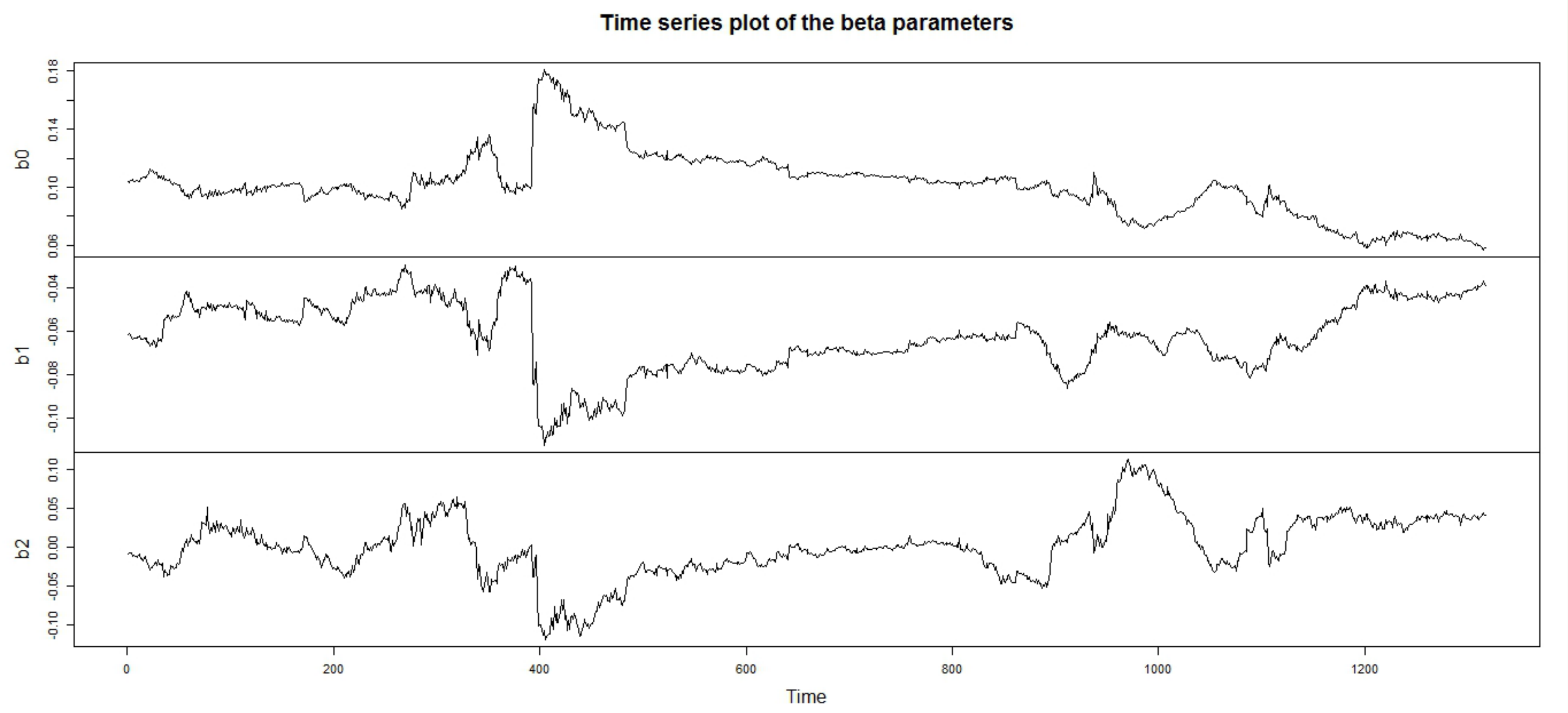
FORECASTING THE BETAS

- The betas are forecasted by fitting univariate time series models on three time series data:
- The three time series of betas:

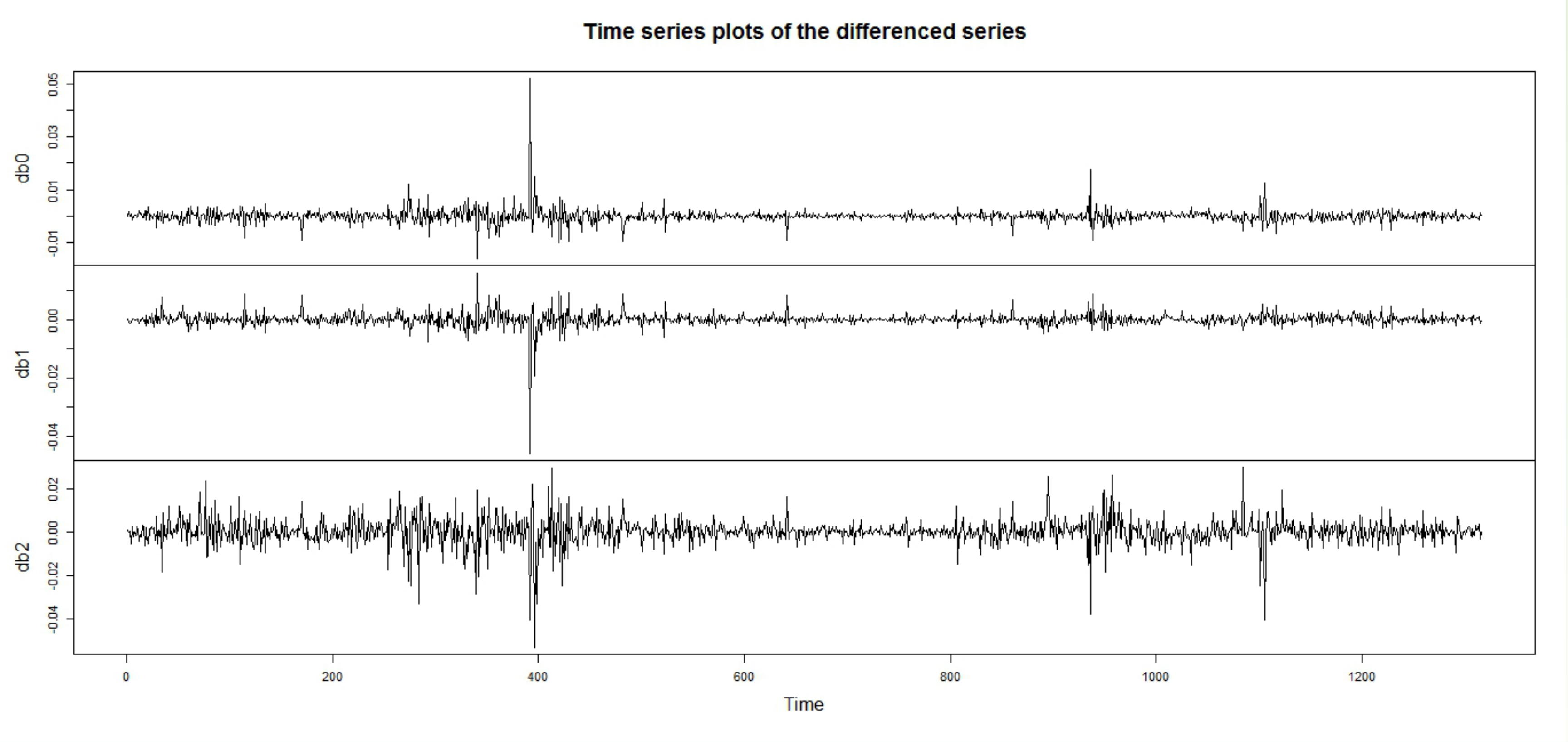
$$\{\beta_{0t}: t \in \{-127, -126, \dots, -1, 0\}\}$$
$$\{\beta_{1t}: t \in \{-127, -126, \dots, -1, 0\}\}$$
$$\{\beta_{2t}: t \in \{-127, -126, \dots, -1, 0\}\}$$

- In order to predict the next betas using the above time series, each of the above time series must be stationary, otherwise differencing would have to be done to obtain a stationary time series.

NON-STATIONARITY OF THE BETAS



FIRST-ORDER DIFFERENCE TIME SERIES OF THE BETAS



FITTED TIME SERIES MODEL FOR β_0

For β_0, we have GARCH(1,1) given by

$$\mathrm{d}\beta_{0,t} = -002723 + \sigma_{0,t} Z_{0,t}$$

$$\left(\sigma_{0,t}\right)^2 = 0.000002689 + 0.4246\left(d\beta_{0,t-1}\right)^2 + 0.3772\left(\sigma_{0,t-1}\right)^2$$

$$\hat{\sigma}_{0,0} = 0.00335072$$

$$\beta_{0,0} = 0.068871$$

$$d\beta_{0,0} = 0.003494$$

$$\beta_{0,1} = \beta_{0,0} + d\beta_{0,1}$$

$$\beta_{0,t+1} = \beta_{0,t} + d\beta_{0,t+1}$$

$$t = 1,2,3, \ldots, 29$$

FITTED TIME SERIES MODEL FOR β_1

For β_1, we have GARCH(1,1) given by

$$\mathrm{d}\beta_{1,t} = 0.0005151 + \sigma_{1,t} Z_{1,t}$$

$$(\sigma_{1,t})^2 = 0.0000007143 + 0.2565(d\beta_{1,t-1})^2 + 0.7338(\sigma_{1,t-1})^2$$

$$\hat{\sigma}_{1,0} = 0.00293082$$

$$\beta_{1,0} = 0.068871$$

$$d\beta_{1,0} = -0.00169$$

$$\beta_{1,1} = \beta_{1,0} + d\beta_{1,1}$$

$$\beta_{1,t+1} = \beta_{1,t} + d\beta_{1,t+1}$$

$$t = 1,2,3,\ldots,29$$

FITTED TIME SERIES MODEL FOR β_2

For β_2, we have GARCH(1,1) given by

$$\mathrm{d}\beta_{2,t} = 0.000298 + \sigma_{2,t} Z_{2,t}$$
$$\left(\sigma_{2,t}\right)^2 = 0.000004241 + 0.3402\left(d\beta_{2,t-1}\right)^2 + 0.6808\left(\sigma_{2,t-1}\right)^2$$
$$\hat{\sigma}_{2,0} = 0.00976302$$
$$\beta_{2,0} = 0.0088305$$
$$d\beta_{2,0} = -0.01073$$

$$\beta_{2,1} = \beta_{2,0} + d\beta_{2,1}$$
$$\beta_{2,t+1} = \beta_{2,t} + d\beta_{2,t+1}$$
$$t = 1,2,3,\ldots,29$$

FITTING A COPULA MODEL FOR THE RESIDUALS $\{Z_{0,t}\}, \{Z_{1,t}\}, \{Z_{2,t}\}$

- Best fit: t copula with correlation matrix

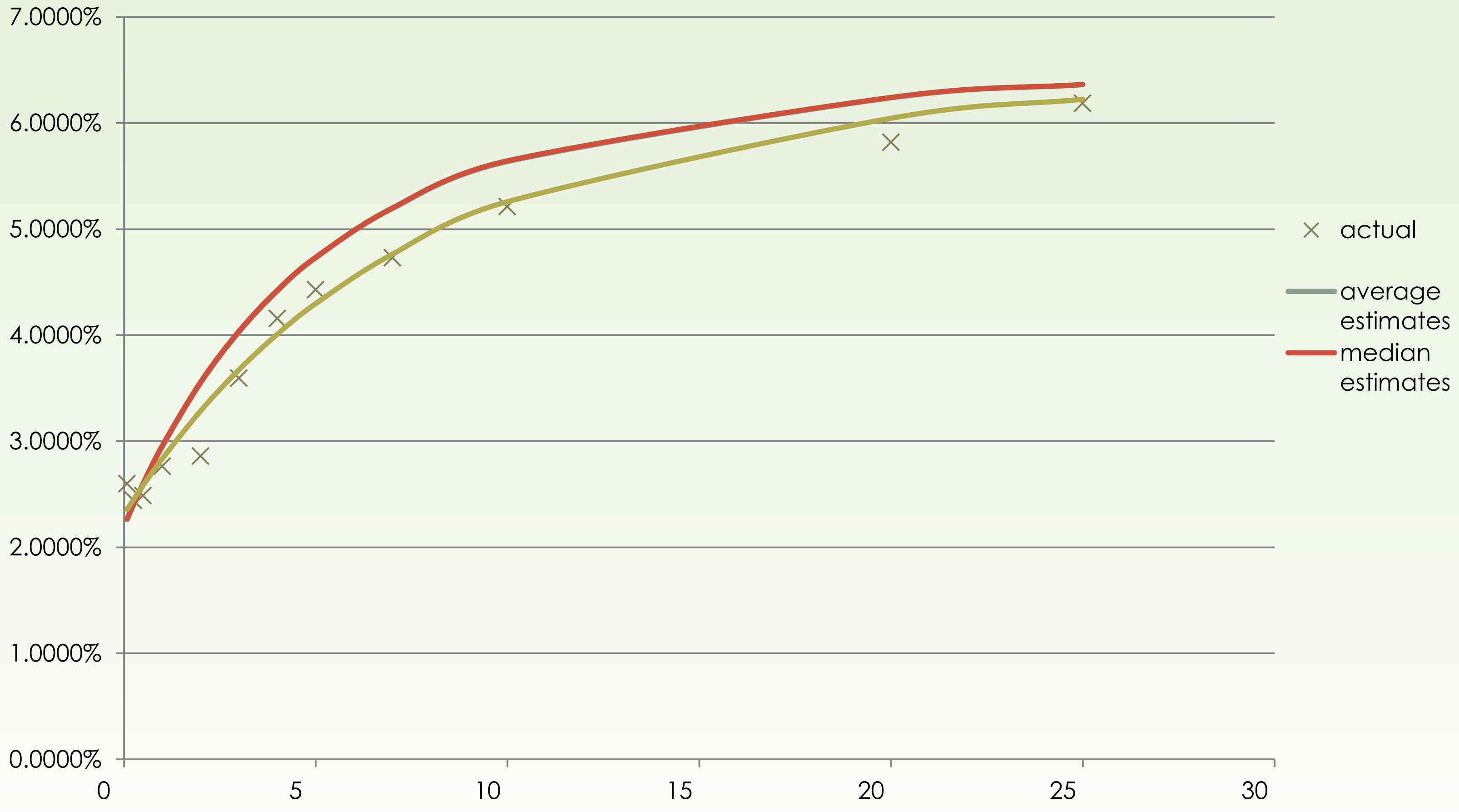
$$\begin{bmatrix} 1 & -0.859 & -0.838 \\ -0.859 & 1 & 0.631 \\ -0.838 & 0.631 & 1 \end{bmatrix}$$

- i.e. $\{Z_{0,t}\}, \{Z_{1,t}\}, \{Z_{2,t}\}$ from actual data were found to be correlated with the given sample correlation matrix .

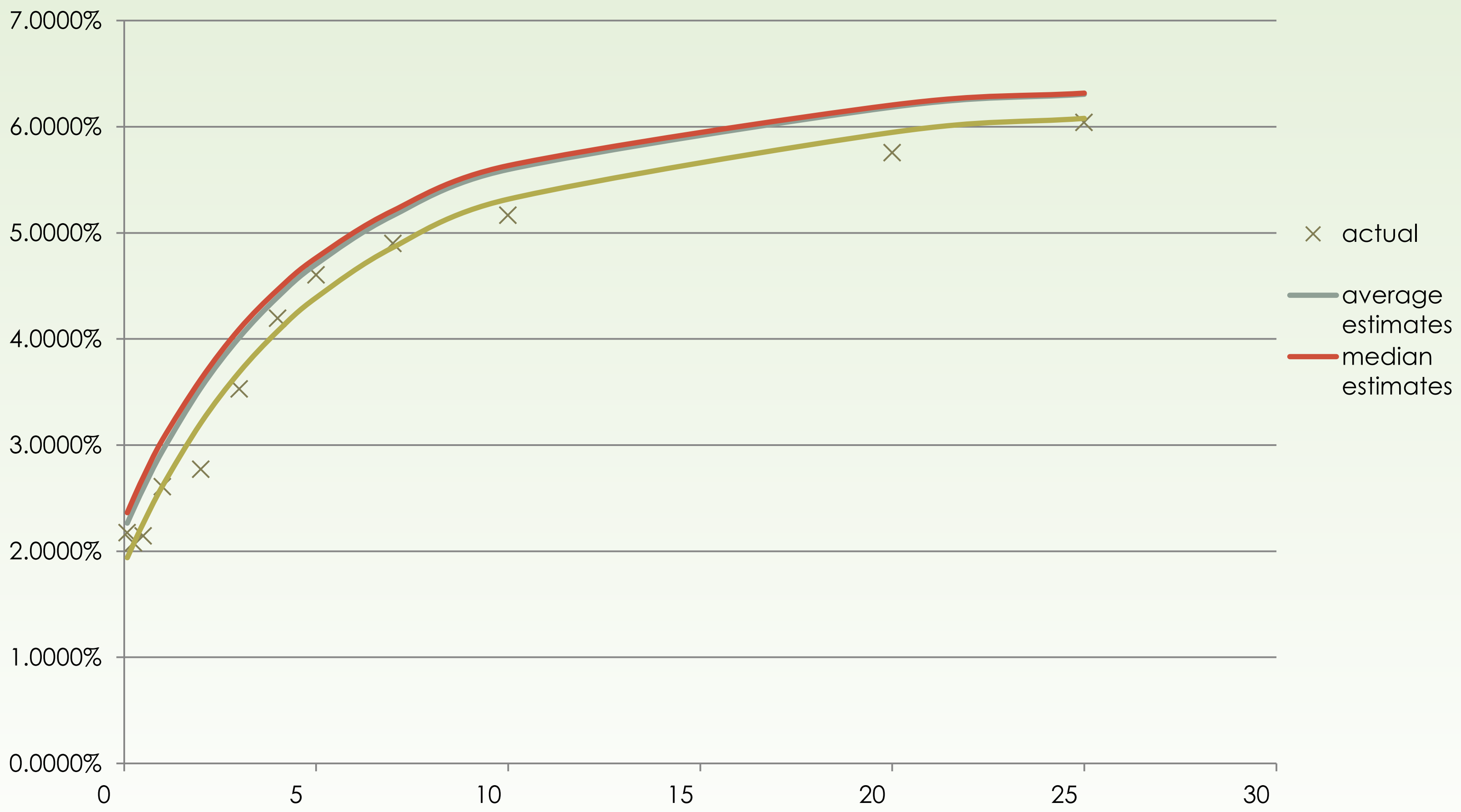
GENERATING $\{Z_{i,t}\}$

- For $t = 1,2,3, \ldots, 30$ (i.e up to 30 weeks ahead from March 28, 2012), $\{Z_{0,t}\}$, $\{Z_{1,t}\}$, $\{Z_{2,t}\}$ were generated from the t copula obtained earlier.
- This generates the 1-week, 2-week, …, , and 30-week ahead forecast of the yield curve.
- This generation is done 10,000 times. So there are 10,000 yield curves generated for 1 week ahead, 10,000 yield curve for 2 weeks ahead, and so on until 30 weeks ahead.
- The median and mean yield curves are constructed from each set of 10,000 yield curves.

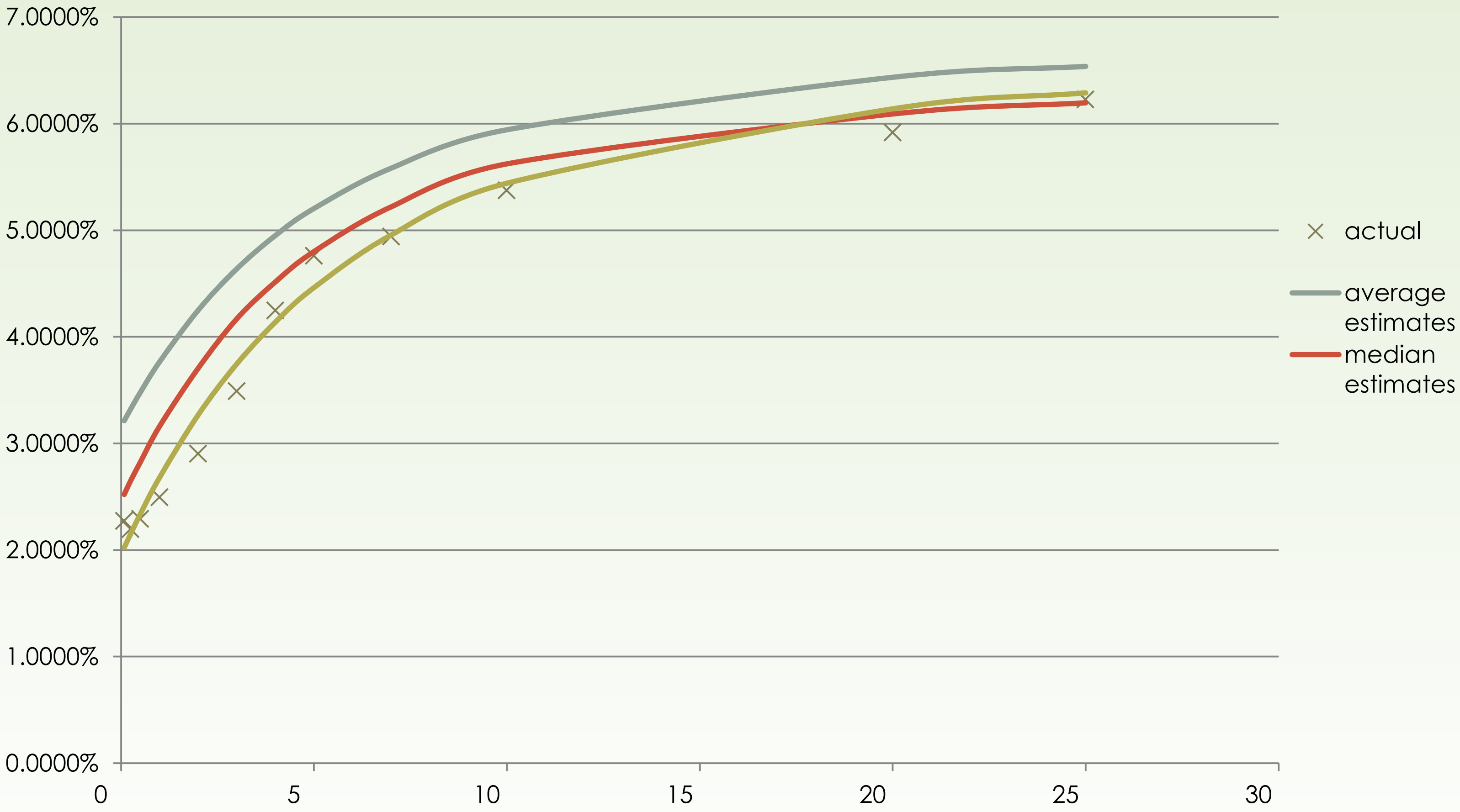
1-WEEK AHEAD SIMULATION COMPARED WITH ACTUAL OBSERVATION



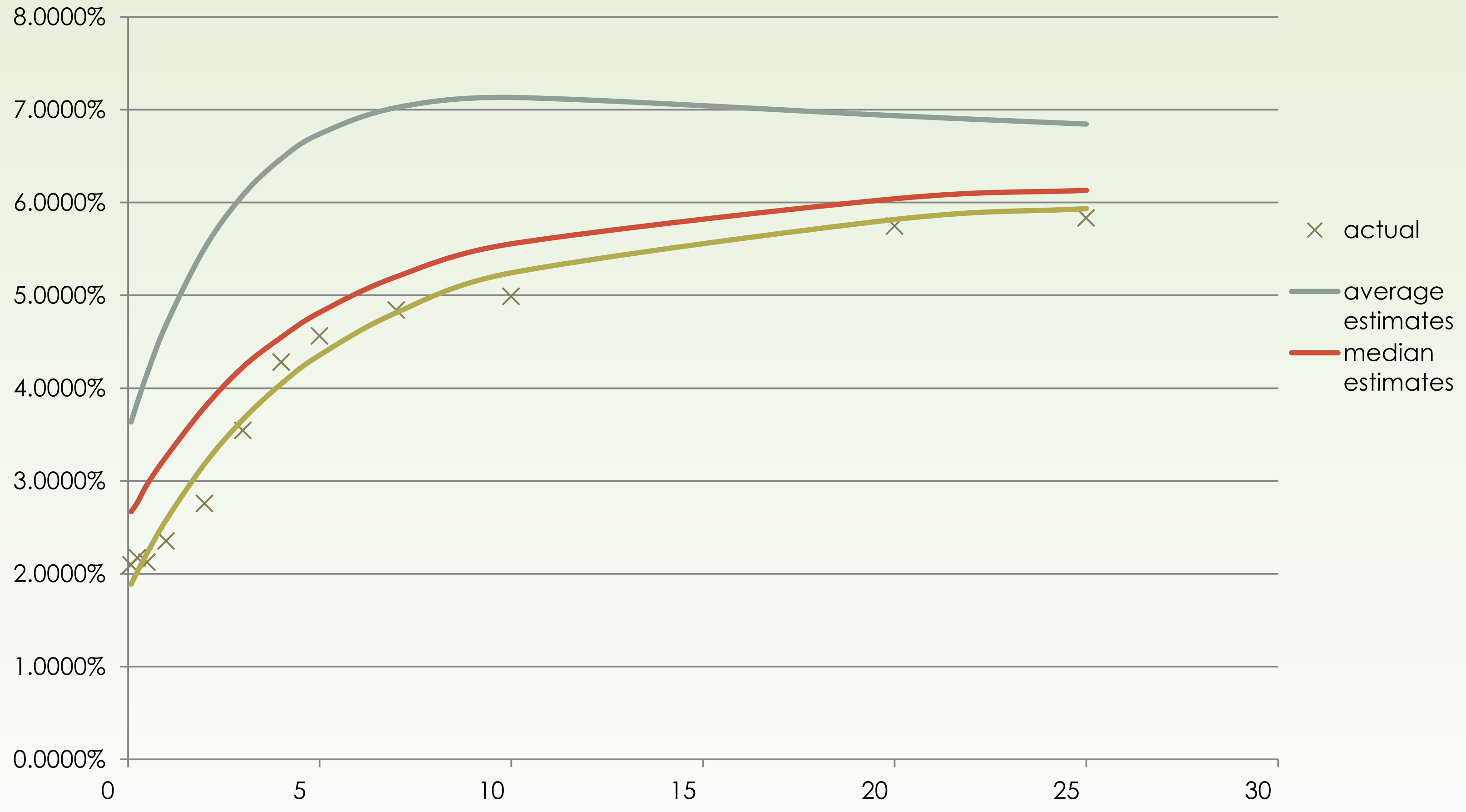
5-WEEK AHEAD SIMULATION COMPARED WITH ACTUAL OBSERVATION



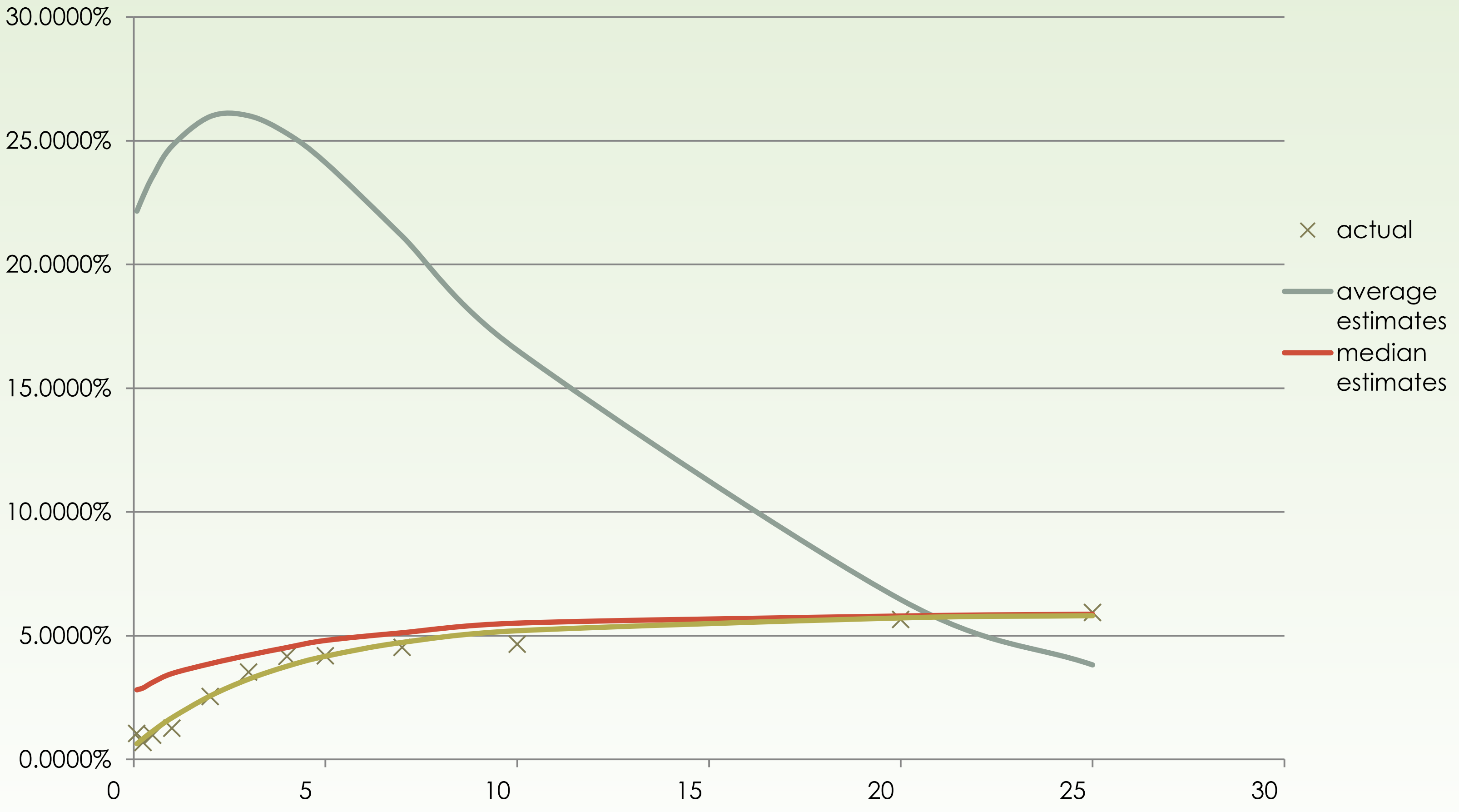
10-WEEK AHEAD SIMULATION COMPARED WITH ACTUAL OBSERVATION



15-WEEK AHEAD SIMULATION COMPARED WITH ACTUAL OBSERVATION



30-WEEK


30.0000%
25.0000%
20.0000%
15.0000%
10.0000%
5.0000%
0.0000%
0
5
10
15
20
25
30
actual
average estimates
median estimates

YIELD CURVE ESTIMATION

- Suppose that today is time $t = 0$.

τ (in years)	Total Coupon per Year (Pesos)	Bond Price	$y_0(\tau)$
0.5	0	98	4.0405%
1	0	95	5.1293%
1.5	6.2	101	5.4429%
2	8	104	5.8085%

- $98 = 100e^{-y_0(0.5)}$
  $95 = 100e^{-y_0(1.0)}$
  $101 = 3.1e^{-y_0(0.5)} + 3.1e^{-y_0(1.0)} + 103.1e^{-y_0(1.5)}$
  $104 = 4e^{-y_0(0.5)} + 4e^{-y_0(1.0)} + 4e^{-y_0(1.5)} + 104e^{-y_0(2.0)}$

YIELD CURVE ESTIMATION