On Integration-by-parts and the Itô Formula for Backwards Itô Integral

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Abstract: In this paper, we derive integration-by-parts formula using the generalized Riemann approach to stochastic calculus called the backwards Itô integral. Moreover, we use integration-by-parts formula to deduce the Itô formula for the backwards Itô integral.

Keywords/phrases: Backwards Itô integral; backwards L^2-martingale; AC^2-property.

1 Introduction

Itô Formula is the chain rule for stochastic calculus. It is one of the most important tools in stochastic calculus. The first version of this fundamental result was proved by Itô [10] in 1951. In this paper, we derived the Itô Formula by proving the integration-by-parts using the generalized Riemann approach to stochastic integrals which is called the backwards Itô integral [4, 3]. This approach is also called the Henstock approach and was discovered independently by J. Kurzweil in 1950s and by R. Henstock in the 1960s when studying the classical (non-stochastic) integral.

The reader is reminded that it is impossible to define stochastic integrals using the Riemann approach since the integrators have paths of unbounded variation and the integrands are highly oscillatory. Therefore, Henstock approach has been used to define the Itô integral instead. See, for instance, [5], [8], [14], [15], [17], [18], and [20] among others. These studies have shown that integrals defined by the generalized Riemann approach encompass the classical stochastic integral.

Recently, Toh, T.L and Chew, T.S. [16, pp.657-660], used the same approach to prove their integration-by-parts Formula for the Itô-Kurzweil-Henstock integral. However, their integral was defined using Riemann sums where the tags are always on the left endpoints of the subintervals. This integral is equivalent to the classical (forward) Itô integral when the integrator is a Brownian motion.

In this paper, backwards Itô integral [4] was defined using generalized Riemann approach, but the δ-fine division used to define our integral is *backwards* in the sense that the tags are the right endpoints of the disjoint left-open subintervals. The adaptedness property is preserved by using backwards filtration. Note that backwards δ-fine division is *partial* since a backwards δ-fine $full$ division may not exist. By using Vitali's Covering theorem, we are assured that a *partial* backwards δ-fine division that covers *almost* the entire interval $[0, T]$ exists, that is, except for a small part of $[0, T]$ whose Lebesgue measure can be made small.

2 Preliminaries

We will assume familiarity with the definitions and basic properties which can be found in [3]. Throughout this note, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_0^+$ the set of nonnegative real numbers, $\mathbb{N}$ the set of positive integers and $(\Omega, \mathcal{G}, \mathbf{P})$ denotes a probability space.

Let $\{\mathcal{G}^s : 0 \le s \le T\}$ be a family of sub σ-algebras of $\mathcal{G}$. Then $\{\mathcal{G}^s : 0 \le s \le T\}$ is called a backwards filtration if

$\mathcal{G}^t \subseteq \mathcal{G}^s$ for all $0 \leq s < t \leq T$. If in addition, $\{\mathcal{G}^s : 0 \leq s \leq T\}$ satisfies the following condition: (1) $\mathcal{G}^T$ contains all sets of $\mathbf{P}$ -measure zero in $\mathcal{G}$; and (2) for each $s \in [0,T]$, $\mathcal{G}^s = \mathcal{G}^{s-}$ where $\mathcal{G}^s = \bigcap_{\varepsilon>0} \mathcal{G}^{s-\varepsilon}$. Then $\{\mathcal{G}^s : 0 \leq s \leq T\}$ is called a *standard backwards filtration.* We often write $\{\mathcal{G}^s\}$ instead of $\{\mathcal{G}^s : 0 \leq s \leq T\}$.

A *stochastic process* f or simply *process* is a function $f : \Omega \times [0,T] \to \mathbb{R}$, where $[0,T]$ is an interval in $\mathbb{R}_0^+$ and $f(\cdot, s)$ is $\mathcal{G}^s$-measurable for each $s \in [0,T]$. A process $f = \{f_s : s \in [0,T]\}$ is said to be *adapted* to the standard backwards filtration $\{\mathcal{G}^s\}$ if f_s is $\mathcal{G}^s$-measurable for each $s \in [0,T]$. Let $B = \left\{B_t : t \in \mathbb{R}_0^+\right\}$ be a standard Brownian motion (BM). Let $\sigma(B_u : s \leq u \leq T)$ be the smallest σ-algebra generated by $\{B_u : s \leq u \leq T\}$. This is the smallest σ-algebra containing the information about the structure of BM on $[s,T]$.

Throughout this note, we assume that the standard backwards filtration $\{\mathcal{G}^s\}$ is the family of σ-algebras $\sigma(B_u : s \leq u \leq T)$. This family is then called the *natural backwards filtration* of B, (see [1, pp. 239-240]). Let $(\Omega, \mathcal{G}, \{\mathcal{G}^s\}, \mathbf{P})$ be a *standard backwards filtering space.* We write $L^p(\Omega)$ for $L^p(\Omega, \mathcal{G}, \mathbf{P})$ and $f \in L^p(\Omega)$ if $\mathbb{E}(f)^p < \infty$. We also define $\|f\|_p = \{\mathbb{E}(|f|^p)\}^{\frac{1}{p}}$. For $f \in L^1(\Omega)$, let $\mathbb{E}(f)$ denote the expectation of f, that is, $\mathbb{E}(f) = \int_\Omega f d\mathbb{P}$. The conditional expectation of f given $\mathcal{G}^s$ is the random variable $\mathbb{E}(f|\mathcal{G}^s)$.

3 Backwards Itô Integral

In this section, we shall present the backwards Itô integral and some related results. The reader is refered to [4, 3] for proof.

Let $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ be a finite collection of interval-point pairs of $(0,T]$. Then D is said to be a partial division of $[0,T]$ if $\{(u_i,\xi_i]\}_{i=1}^n$ are disjoint subintervals of $(0,T]$. In addition, if $\bigcup_{i=1}^n(u_i,\xi_i]=(0,T]$ then D is a division of $[0,T]$.

Let δ be a positive function on $(0,T]$. An interval-point pair $((u,\xi],\xi)$ is said to be backwards δ-fine if $(u,\xi]\subseteq(\xi-\delta(\xi),\xi]$, whenever $(u,\xi]\subseteq[0,T]$ and $\xi\in(0,T]$. We call D a backwards δ-fine partial division of $[0,T]$ if D is a partial division of $[0,T]$ and for each i, $((u_i,\xi_i],\xi_i)$ is backwards δ-fine. Given $\varepsilon>0$, a backwards δ-fine partial division D is said to fail to cover $(0,T]$ by at most Lebesgue measure η if $|T-\sum_{i=1}^n(\xi_i-u_i)|\leq\eta$.

We remark that given any positive function δ, one may not be able to find a (full) division that covers the entire interval $(0,T]$. For example, take $\delta(\xi)=\xi/2$. Then the interval $(0,T]$ is not covered by any finite collection of backwards δ-fine intervals.

Definition 3.1 Let $f=\{f_s:s\in[0,T]\}$ be a process adapted to the standard backwards filtering space $(\Omega,\mathcal{G},\{\mathcal{G}^s\},\mathbf{P})$. Then f is said to be backwards Itô integrable on $[0,T]$ if there exists an $A\in L^2(\Omega)$ such that for every $\varepsilon>0$, there exist a positive function $\delta(\xi)$ on $[0,T]$ and a positive number η such that for any backwards δ-fine partial division $D=\{((u_i,\xi_i],\xi_i)\}_{i=1}^n$ of $[0,T]$ that fails to cover $[0,T]$ by at most Lebesgue measure η we have

$$\mathbb{E}\Big(|\sum_{i=1}^n f_{\xi_i}\left(B_{\xi_i}-B_{u_i}\right)-A|^2\Big)\leq\varepsilon.$$

We call A the *backwards Ito integral of* f, that is, $A=\int_0^T f_t\,dB_t$. It is not difficult to check that the integral A is unique up to a set of $\mathbf{P}$-measure zero.

Example 3.2 The Brownian motion B is backwards Itô integrable on $[a,b]$ with respect to itself, and

$$\int_a^b B_t dB_t=\frac{1}{2}\left(B_b^2-B_a^2\right)+\frac{1}{2}(b-a).$$

The following definitions and theorems can be found in [4, 3].

Definition 3.3 [2, p.509] Let $F = \{F_s : s \in [0,T]\}$ be a stochastic process. Then the process F is said to have an AC^2- *property* if for each $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\mathbb{E}\Big(\big|\sum_{i=1}^{n} F(u_i, v_i)\big|^2\Big) \leq \varepsilon,$$

for any finite collection of disjoint subintervals $\{(u_i, v_i]\}_{i=1}^{n}$ of $[0,T]$ for which

$$\sum_{i=1}^{n} |v_i - u_i| \leq \eta.$$

Theorem 3.4 *Let f be backwards Itô integrable on $[0,T]$. Let*
$\Phi(\xi, T) = \int_{\xi}^{T} f_t dB_t$. Then Φ has AC^2-property.

Theorem 3.5 *Let f and F be stochastic processes on $[0,T]$. Then f is backwards Itô integrable to F on $[0,T]$ if and only if (i) F satisfies AC^2-property and (ii) for every $\varepsilon > 0$, there exists a positive function δ on $[0,T]$ such that for every backwards δ-fine partial division $D = \{((u, \xi], \xi)\}$ of $[0,T]$ we have,*

$$\mathbb{E}\Big(\big|(D)\sum_{i=1}^{n}(B_{\xi_i} - B_{u_i}) - F(u_i, \xi_i)\big|^2\Big) \leq \varepsilon.$$

Definition 3.6 An adapted process $f = \{f_s : s \in [0,T]\}$ on $(\Omega, \mathcal{G}, \{\mathcal{G}^s\}, \mathbf{P})$ is called backwards L^2- martingale if

(*i*) $\mathbb{E}(|f_s|) < \infty$, for all $s \in [0,T]$;

(*ii*) $\mathbb{E}(f_t|G^s) = f_s$ whenever $0 \leq t \leq s \leq T$; and (*iii*) $\sup_{t\in[0,T]}\int_\Omega |f_t|^2 dP < \infty$.

Theorem 3.7 *Let f be backwards Itô integrable on any subinterval $[0,T]$ of $[0,\infty)$ with $F(s,T) = \int_s^T f_t dB_t$. Then the process $\{F_s : s \in [0,T]\}$ is backwards L^2-martingale with respect to its natural backwards filtration $\{\mathcal{G}^s\}$.*

We restate Definition 3.1 in the light of Theorem 3.4 and Theorem 3.5 as follows.

Definition 3.8 Let $f = \{f_t\}_{t\in[0,T]}$ be backwards adapted process on $(\Omega, \mathcal{G}, \{\mathcal{G}^t\}, \mathbf{P})$. Then f is said to be *backwards Itô integrable on* $[0,T]$ if there exists a backwards L^2-martingale process $F = \{F_t\}_{t\in[0,T]}$ that satisfies the AC^2-property on $[0,T]$ such that for every $\varepsilon > 0$, there exists a positive function $\delta(\xi) > 0$ such that for every backwards δ-fine partial division $D = \{((u_i,\xi_i],\xi_i)\}_{i=1}^n$ of $[0,T]$ we have

$$\mathbb{E}\left(\sum_{i=1}^n \left| f_{\xi_i}(B_{\xi_i} - B_{u_i}) - (F_{\xi_i} - F_{u_i})\right|^2\right) \le \varepsilon.$$

In brief, we denote $F_{\xi_i} - F_{u_i}$ by $F(u_i, \xi_i)$.

4 Integration-By-Parts for the Backwards Itô Integral

In this section, we shall establish the integration-by-parts and prove some results relating to it.

Definition 4.1 Let $G = \{G_t : t \in [0,T]\}$ be a backwards process. For every backwards δ-fine partial division $D = \{((u_i,\xi_i],\xi_i)\}_{i=1}^n$ of $[0,T]$, we define

$$\Delta G[u,\xi] := G_\xi - G_u = G(u,\xi) \tag{1}$$

and

$$V(\Delta G) = \inf_{\delta} \sup_{D} \sum_{i=1}^{n} \mathbb{E}\Big(G(u_i, \xi_i)\Big)^2 \tag{2}$$

where sup is taken over all possible backwards δ-fine partial division D of $[0, T]$ and inf is taken over all possible positive function $\delta(\xi)$ on $[0, T]$.

Definition 4.2 Let $f = \{f_t\}_{t\in[0,T]}$ be backwards adapted process on $(\Omega, \mathcal{G}, \{\mathcal{G}^t\}, \mathbf{P})$. In Definition 3.8, replacing B by a backwards adapted process $X = \{X_t\}_{t\in[0,T]}$, an integral $\int_0^T f dX$ is defined, which is also called *the backwards Itô integral with respect to* X. The stochastic process $F(t, T) = \int_t^T f dX$ is not necessarily backwards L^2-martingale.

Remark 4.3 $V(\Delta G) = 0$ if and only if for every $\varepsilon > 0$, there is a $\delta(\xi) > 0$ on $[0, T]$ such that for every backwards δ-fine partial division $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ of $[0, T]$ we have

$$(D) \sum_{i=1}^{n} \mathbb{E}\left(G\left(u_i, \xi_i\right)\right)^2 \le \varepsilon.$$

The following result is very crucial in proving the Itô's Formula.

Theorem 4.4 (*Integration-By-Parts*). *Let f and X be backwards adapted defined on a standard backwards filtering space $(\Omega, \mathcal{G}, \{\mathcal{G}^t\}, \mathbf{P})$. Assume that f is backwards Itô integrable with respect to X on $[0, T]$. Then there exists a backwards adapted process H such that $V(\Delta f \Delta X - \Delta H) = 0$ if and only if X is backwards Itô integrable with respect to f on $[0, T]$. Moreover,*

$$\int_0^T f(s)dX(s) + \int_0^T X(s)df(s) = f(T)X(T) - f(0)X(0) + (H(T) - H(0)).$$

Proof: Here, we denote $f(\xi) = f_\xi$, $X(\xi) = X_\xi$ and $H(\xi) = H_\xi$. Suppose a backwards adapted process H exists. Let $F(\xi, T) = \int_\xi^T f(t)dX(t)$ and define

$$\begin{aligned} G(\xi,T) &= f(T)X(T) - f(\xi)X(\xi) - \int_\xi^T f(t)dX(t) \\ &\quad + (H(T) - H(\xi)) \\ &= f(T)X(T) - f(\xi)X(\xi)) - F(\xi,T) + H(\xi,T). \end{aligned}$$

Given any $\varepsilon > 0$, there exists a positive function δ_1 on $[0,T]$ such that for every backwards δ_1-fine partial division $D_1 = \{((u_i, \xi_i], \xi_i)\}_{i=1}^{n_1}$ of $[0,T]$ we have,

$$\sum_{i=1}^{n_1} \mathbb{E}\Big(f_{\xi_i}(X_{\xi_i} - X_{u_i}) - F(u_i, \xi_i) \Big)^2 \le \frac{\varepsilon}{4}.$$

Since $V(\Delta f \Delta X - \Delta H) = 0$ it follows by Remark 4.3, that there exists a positive function δ_2 on $[0,T]$ such that for every backwards δ_2-fine partial division $D_2 = \{((u_i, \xi_i], \xi_i)\}_{i=1}^{n_2}$ of $[0,T]$ we have,

$$\sum_{i=1}^{n_2} \mathbb{E}\Big((f_{\xi_i} - f_{u_i})(X_{\xi_i} - X_{u_i}) - H(u_i, \xi_i) \Big)^2 \le \frac{\varepsilon}{4}.$$

Define $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$ for all $\xi \in [0,T]$. Let $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^{n}$ be a backwards δ-fine partial division of $[0,T]$. Then we have,

$$\sum_{i=1}^{n} \mathbb{E}\Big(X_{\xi_i}\,(f_{\xi_i} - f_{u_i}) - G(u_i, \xi_i) \Big)^2 < \varepsilon.$$

Hence, X is backwards Itô integrable with respect to f on $[0,T]$.

Conversely, suppose that X is backwards Itô integrable with respect to f on $[0,T]$ with $K(t,T) = \int_t^T X(s)df(s)$. We follow the idea above by letting

$$H(\xi,T) = \int_\xi^T f(t)dX(t) + \int_\xi^T X(t)df(t) \\ - \left[f(T)X(T) - f(\xi)X(\xi)\right]. \tag{3}$$

Then given any $\varepsilon > 0$, there exists a positive function δ_1 on $[0,T]$ such that for every $D_1 = \{((u_i,\xi_i],\xi_i)\}_{i=1}^{n_1}$ backwards δ_1-fine partial division of $[0,T]$ we have,

$$\sum_{i=1}^{n_1} \mathbb{E}\Big(X_{\xi_i}(f_{\xi_i} - f_{u_i}) - (K_{\xi_i} - K_{u_i})\Big)^2 \le \frac{\varepsilon}{4}.$$

Note that f is backwards Itô integrable to

$$F(t,T) = \int_t^T f(s)dX(s)$$

with respect to X on $[0,T]$. For each $\varepsilon > 0$, there exists a positive function δ_2 on $[0,T]$ such that for every $D_2 = \{((u_i,\xi_i],\xi_i)\}_{i=1}^{n_2}$ backwards δ_2-fine partial division of $[0,T]$ we have,

$$\sum_{i=1}^{n_2} \mathbb{E}\Big(f_{\xi_i}(X_{\xi_i} - X_{u_i}) - (F_{\xi_i} - F_{u_i})\Big)^2 \le \frac{\varepsilon}{4}.$$

We will show that $V(\Delta f \Delta X - \Delta H) = 0$, that is, for each $\varepsilon > 0$, there exists a positive function δ on $[0,T]$ such that for every backwards δ-fine partial division $D_0 = \{((u_i,\xi_i],\xi_i)\}_{i=1}^{n}$ of $[0,T]$ we have,

$$\sum_{i=1}^{n} \mathbb{E}\Big((f_{\xi_i} - f_{u_i})(X_{\xi_i} - X_{u_i}) - (H_{\xi_i} - H_{u_i})\Big)^2 \le \frac{\varepsilon}{4}.$$

Now, define $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$ for all $\xi \in [0, T]$. Let $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ be a backwards δ-fine partial division of $[0, T]$. Then we have,

$$\sum_{i=1}^{n} \mathbb{E}\Big((f_{\xi_i} - f_{u_i})(X_{\xi_i} - X_{u_i}) - H(u_i, \xi_i)\Big)^2 \leq \varepsilon.$$

The proof is now complete. □

The following corollary is a special case of Theorem 4.4 by putting $H(t) \equiv 0$.

Corollary 4.5 *Let f and X be backwards adapted processes defined on $(\Omega, \mathcal{G}, \{\mathcal{G}^t\}, \mathbf{P})$. If f backwards Itô integrable with respect to X on $[0, T]$ and $V(\Delta f \Delta X) = 0$, then X is backwards Itô integrable with respect to f on $[0, T]$. Furthermore,*

$$\int_0^T f(s)dX(s) + \int_0^T X(s)df(s) = f(T)X(T) - f(0)X(0).$$

Also, by letting $f = X$, the next corollary follows.

Corollary 4.6 *Let X be backwards adapted processes defined on the standard backward filtering space $(\Omega, \mathcal{G}, \{\mathcal{G}^t\}, \mathbf{P})$ such that*

$$V(\Delta X)^2 = 0.$$

Then X is backwards Itô integrable with respect to X and

$$\int_0^T X(s)dX(s) = \frac{1}{2}X(T)^2 - \frac{1}{2}X(0)^2.$$

The next result follows from Theorem 4.4 by setting $f = X$ and $H = G$.

Corollary 4.7 *Let X be backwards adapted processes defined on the standard backward filtering space $(\Omega, \mathcal{G}, \{\mathcal{G}^t\}, \mathbf{P})$ such that there exists a process G with*

$$V((\Delta X)^2 - \Delta G)^2 = 0.$$

Then X is backwards Itô integrable with respect to X and

$$\int_0^T X(s)dX(s) = \frac{1}{2}X(T)^2 - \frac{1}{2}X(0)^2 + \frac{1}{2}\left(G(T) - G(0)\right).$$

Lemma 4.8 *The following are true; (i) $V((\Delta B)^2 - \Delta G)^2 = 0$, where $G(u,\xi) = \xi - u$, and (ii.) For each $\varepsilon > 0$, there exists a positive function δ on $[0,T]$ such that for every backwards δ-fine partial division $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ of $[0,T]$, we have*

$$\sum_{i=1}^n \left[\mathbb{E}\left((B_{\xi_i} - B_{u_i})^2 - (\xi_i - u_i)\right)^4\right]^{\frac{1}{2}} \leq \varepsilon.$$

Proof: To prove (i), let $\varepsilon > 0$ be given and choose $\delta(\xi) = \frac{\varepsilon}{2T}$ so that for any backwards δ-fine partial division $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ of $[0,T]$ we have

$$\sum_{i=1}^n \mathbb{E}\left((B_{\xi_i} - B_{u_i})^2 - G(u,\xi)\right)^2 \leq \varepsilon.$$

In proving (ii), we need to use the fact that

$$E\Big((B_{\xi_i} - B_{\xi_i})\Big)^{2p} = C_p(\xi_i - u_i)^p, \tag{4}$$

for some constant C_p, $p > 0$ (see [19, p.283]). The proof is similar to that of (i). □

Example 4.9 Let X be a Brownian motion B which is by definition is backwards adapted to standard backwards filtering space $(\Omega, \mathcal{G}, \{\mathcal{G}^t\}, \mathbf{P})$. By applying the Theorem 4.7 and Lemma 4.8(i), B is backwards Itô integrable with respect to itself. Moreover, by definition of G in Lemma 4.8(i) we have

$$\int_0^T B(s)dB(s) = \frac{1}{2}B(T)^2 - \frac{1}{2}B(0)^2 + \frac{1}{2}T \tag{5}$$

Observe that there is one extra term, the $\frac{1}{2}T$. This is because the Brownian motion is of unbounded variation.

5 The Itô Formula

The integration-by-parts formula that is developed earlier is the main tool in proving the Itô Formula. It asserts that, if $F : \mathbb{R} \to \mathbb{R}$ is a real valued function which is at least twice continuously differentiable on $\mathbb{R}$ and that the following condition (Δ) holds:

(1) $\displaystyle\int_0^T F'(B(t))dB(t)$, and;

(2) $\displaystyle\int_0^T F''(B(t))dB(t)$

both exist in terms of L^2-convergence.

The goal is to show that

$$\begin{aligned}\int_0^T F'(B(t))dB(t) &= F(B(T)) - F(B(0)) \\ &\quad + \frac{1}{2}\int_0^T F''(B(t))dt\end{aligned} \tag{6}$$

Note that in [11, pp. 105-107], the Itô Formula is shown using convergence in probability but we do not impose such a condition here.

Lemma 5.1 *Let $F : \mathbb{R} \to \mathbb{R}$ be at least twice continuously differentiable with the condition (Δ) above. Then there exists a backwards adapted process H such that $V(\Delta F'\Delta B - \Delta H) = 0$ if and only if the backwards adapted process $B(t)$ is backwards Itô integrable with respect to $F'(B(t))$ on $[0, T]$. Furthermore,*

$$\begin{aligned}\int_0^T F'(B(t))dB(t) + \int_0^T B(t)dF'(B(t)) \\ = F'(B(T))B(T) - F'(B(0))B(0) \\ + (H(1) - H(0)).\end{aligned}$$

Proof: Let $f(t) = F'(B(t))$ and $X = B$, and by applying integration-by-parts, the result follows. □

Lemma 5.2 *Let f be backwards Itô integrable with respect to B with the primitive process $\Phi(t, T) = \int_t^T f(s)dB(s)$. Let $G = \{G_t : t \in [0, T]\}$ be backwards adapted process. If $V(\Delta G - \Delta\Phi) = 0$, then for every $\varepsilon > 0$ there exists a positive function δ on $[0, T]$ such that for every backwards δ-fine partial division $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ of $[0, T]$, we have*

$$\sum_{i=1}^{n} \mathbb{E}\Big(f_{\xi_i}(B_{\xi_i} - B_{u_i}) - G(u_i, \xi_i)\Big)^2 \leq \varepsilon.$$

That is, $\Phi(u_i, \xi_i)$ can be replaced by $f_{\xi_i}(B_{\xi_i} - B_{u_i})$ in $V(\Delta G - \Delta\Phi) = 0$.

Proof: Let $\varepsilon > 0$. Since f is backwards Itô integrable with respect to B with the primitive process $\Phi(t, T) =$

$\int_t^T f(s)dB(s)$, it follows that there exists a positive function δ_1 on $[0,T]$ such that for every backwards δ_1-fine partial division $D_1 = \{((u_i,\xi_i],\xi_i)\}_{i=1}^{n_1}$ of $[0,T]$, we have

$$(D_1)\sum_{i=1}^{n_1}\mathbb{E}\Big(f_{\xi_i}(B_{\xi_i}-B_{u_i})-\Phi(u_i,\xi_i)\Big)^2 \leq \frac{\varepsilon}{4}.$$

Since $V(\Delta G - \Delta\Phi) = 0$, it follows from Remark 4.3 that there exists a positive function δ_2 on $[0,T]$ such that for every backwards δ_2-fine partial division $D_2 = \{((u_i,\xi_i],\xi_i)\}_{i=1}^{n_2}$ of $[0,T]$, we have

$$(D_2)\sum_{i=1}^{n_2}\mathbb{E}\Big(G(u_i,\xi_i)-\Phi(u_i,\xi_i)\Big)^2 \leq \frac{\varepsilon}{4}.$$

Define $\delta(\xi) = \min\{\delta_1(\xi),\delta_2(\xi)\}$ for all $\xi\in[0,T]$. Let $D = \{((u_i,\xi_i],\xi_i)\}_{i=1}^{n}$ be a backwards δ-fine partial division of $[0,T]$, then we have

$$(D)\sum_{i=1}^{n}\mathbb{E}\Big(f_{\xi}(B_{\xi_i}-B_{u_i})-G(u_i,\xi_i)\Big)^2 \leq \varepsilon,$$

thereby completing the proof. □

Theorem 5.3 *Let F be a backwards adapted process that is at least twice continuously differentiable with $\displaystyle\int_0^T F'(B(t))dB(t)$ and $\displaystyle\int_0^T F''(B(t))dB(t)$, both exist in terms of L^2-convergence. Furthermore, we assume that*

$$\lim_{t\to s}\mathbb{E}(F''(B(s))-F''(B(t)))^4 = 0.$$

If H is the backwards adapted process given in Lemma 5.1. Then H equals

$$\begin{aligned}H(t,T) = &-\left[F'(B(T))B(T) - F'(B(t))B(t)\right]\\ &+\left[F(B(T)) - F(B(t))\right]\\ &+\frac{1}{2}\int_t^T F''(B(s))ds + \int_t^T B(s)dF'(B(s)).\end{aligned}$$

The following results are needed.

Lemma 5.4 *Let F be a backwards adapted process that is at least twice continuously differentiable. Then for every $\varepsilon > 0$ given, there exists a positive function δ on $[0,T]$ small enough such that when $\theta \in (u,\xi] \subseteq (\xi - \delta(\xi), \xi]$,*

$$\mathbb{E}\left(F''(B(\xi)) - F''\left(B(\theta)\right)\right)^4 \le \varepsilon.$$

Proof: Observe that F'' is continuous and Brownian motion B has continuous sample paths. Therefore, $F''B$ is uniformly continuous on $[0,T]$. Thus, for every $\varepsilon > 0$ there exists a positive function $\delta(\xi)$ on $[0,T]$ small enough such that when $\theta, \xi \in (u,\xi] \subseteq (\xi - \delta(\xi), \xi]$, we have

$$\left\|F''(B(\xi)) - F''\left(B(\theta)\right)\right\|_p \le \varepsilon.$$

By taking $p = 4$ we have,

$$\left\|F''(B(\xi)) - F''\left(B(\theta)\right)\right\|_4 \le \varepsilon,$$

and the result follows. □

Lemma 5.5 *Let F be a backwards adapted process that is at least twice continuously differentiable. Let $D = \{((u_i,\xi_i],\xi_i)\}_{i=1}^n$ be a backwards δ-fine partial division of $[0,T]$. Then for every $\varepsilon > 0$, there exists a positive function $\delta(\xi)$ such that*

$$(D)\sum_{i=1}^n \mathbb{E}\left(F''(B(\xi_i) - F''(B(\theta_i)))\left(B(\xi_i) - B(u_i)\right)^2\right)^2 \le \varepsilon.$$

Proof: Use the fact that $\mathbb{E}(B(\xi_i) - B(u_i))^8 = \beta(\xi_i - u_i)^4$ for some constant β and using Lemma 5.5 the result follows. □
We are now ready to prove the Itô formula.

Proof: Observe that

$$\begin{aligned} H(u,\xi) = F'(B(u))B(u) - F'(B(\xi))B(\xi) \\ + F(B(\xi)) - F(B(u)) \\ + \frac{1}{2}\int_u^{\xi} F''(B(s))ds + \int_u^{\xi} B(s)dF'(B(s)). \end{aligned} \tag{7}$$

In view of Lemma 5.2, in $H(u,\xi)$,

$$\int_u^{\xi} F''(B(s))ds$$

and

$$\int_u^{\xi} B(s)dF'(B(s))$$

can be replaced by $F''(B(\xi))(\xi - u)$ and $B(\xi)(F'(B(\xi)) - F'(B(u)))$ respectively. Therefore,

$$\begin{aligned} \mathbb{E}\left[H(u,\xi)\right]^2 = \mathbb{E}\Big[F'(B(u))B(u) - F'(B(\xi))B(\xi) + F(B(\xi)) \\ - F(B(u)) + \tfrac{1}{2}F''(B(\xi))(\xi - u) \\ + B(\xi)\left(F'\left(B(\xi)\right) - F'\left(B(u)\right)\right)\Big]^2. \end{aligned}$$

On the other hand, apply the Taylor's expansion to F at $B(u)$ up to just second order we get,

$$\begin{aligned} F\left(B(u)\right) = F(B(\xi)) + F'(B(\xi))\left(B(u) - B(\xi)\right) \\ + \frac{1}{2}F''\left(B(\theta)\right)(B(\xi) - B(u))^2 \end{aligned} \tag{8}$$

where $\theta \in (u,\xi]$ and equation (8) holds pathwise. Observe that there exists α between $B(u)$ and $B(\xi)$ such that equation (8) holds with $B(\theta)$ replaced by α. Note also that B

is continuous on $(u, \xi]$, so there exist θ such that $B(\theta) = \alpha$ Furthermore, note that $B(\xi)$ is in fact $B(\omega, \xi)$, that is, $B(\xi)$ depends on $\omega \in \Omega$, so θ also depends on $\omega \in \Omega$. Now by equation (8) we get

$$\begin{aligned}\mathbb{E}\left[H(u,\xi)\right]^2 = \mathbb{E}\Big\{ & F'(B(u))\left[B(u)-B(\xi)\right] + F(B(\xi)) - \\ & \Big\{F(B(\xi)) + F'(B(\xi))(B(u)-B(\xi)) \\ & + \frac{1}{2}F''(B(\theta))(B(\xi)-B(u))^2\Big\} + \\ & \frac{1}{2}F''(B(\xi))(\xi-u)\Big\}^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbb{E}\left[\left(F'(B(\xi)) - F'(B(u))\right)\left(B(\xi)-B(u)\right) - H(u,\xi)\right]^2 \\ &= \mathbb{E}\Big[\frac{1}{2}F''\left(B(\theta)\right)\left(B(\xi)-B(u)\right)^2 - \\ &\tfrac{1}{2}F''(B(\xi))(\xi-u)\Big]^2.\end{aligned} \tag{9}$$

Observe that in equation (9) we assume that $B(\theta)$ be replaced by $B(\xi)$. Now, by Lemma 5.5,

$$(D)\sum_{i=1}^{n}\mathbb{E}\left(F''(B(\xi_i) - F''(B(\theta_i)))\left(B(\xi_i)-B(u_i)\right)^2\right)^2 \leq \varepsilon,$$

whenever $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n$ is a backwards δ-fine partial division of $[0, T]$. Now going back to equation (9), we have

$$\begin{aligned}&\mathbb{E}\left[(F'(B(\xi)) - F'(B(u)))(B(\xi)-B(u)) - H(u,\xi)\right]^2 \\ &= \mathbb{E}\left[\frac{1}{2}F''\left(B(\xi)\right)\left[(B(\xi)-B(u))^2 - (\xi-u)\right]\right]^2\end{aligned} \tag{10}$$

Let $\Gamma_k = \{\xi \in [0,T] : |\mathbb{E}(F''(B(\xi)))^4| \in [k-1,k)\},\ k \in \mathbb{N}$. Then for every $\varepsilon > 0$ and $k \in \mathbb{N}$ by Lemma 4.8(ii) there exists a positive function δ_k on Γ_k such that

$$\sum_{i=1}^{n}\left[\mathbb{E}\left((B(\xi)-B(u))^2-(\xi-u)\right)^4\right]^{\frac{1}{2}} \leq \frac{\varepsilon}{2^k(\sqrt{k})}$$

for any backwards δ_k-fine partial division $D=\{((u_i,\xi_i],\xi_i)\}_{i=1}^{n}$ with $\xi_i \in \Gamma_k$.

Clearly, we have $[0,T]=\bigcup_{k=1}^{\infty}\Gamma_k$. Define $\delta(\xi)=\delta_k(\xi)$, if $\xi \in \Gamma_k$, for $k=1,2,3,\ldots$. Let $D=\{((u_i,\xi_i],\xi_i)\}_{i=1}^{n}$ be any backwards δ-fine partial division of $[0,T]$, and $D_k = \{(u_i,\xi_i],\xi_i)\in D : \xi_i \in \Gamma_k\}$. Then

$$\begin{aligned}
&(D)\sum_{i=1}^{n}\mathbb{E}\left((F''(B(\xi_i)))^2\left((B_{\xi_i}-B_{u_i})^2-(\xi_i-u_i)\right)^2\right)\\
&\leq (D_k)\sum_{i}\left[\left(\mathbb{E}\left(F''\left(B\left(\xi_i\right)\right)\right)^4\right]\left[\mathbb{E}\left(B_{\xi_i}-B_{u_i}\right)^2-\left(\xi_i-u_i\right)^4\right]^{\frac{1}{2}}\\
&\leq \sum_{i=1}^{\infty}\sqrt{k}(D_k)\sum_{i}\mathbb{E}[((B_{\xi_i}-B_{u_i})^2-(\xi_i-u_i)^4]^{\frac{1}{2}}\\
&\leq \sum_{i=1}^{\infty}\sqrt{k}\frac{\varepsilon}{2^k(\sqrt{k})}\\
&= \sum_{i=1}^{\infty}\frac{\varepsilon}{2^k}\\
&= \varepsilon.
\end{aligned} \tag{11}$$

Hence, by equation (10), we have

$$\begin{aligned}
&(D)\sum_{i=1}^{n}\mathbb{E}\left(\left(F'(B(\xi))-F'(B(u))\right)(B(\xi)-B(u))-H(u,\xi)\right)^2\\
&=\frac{1}{4}(D)\sum_{i=1}^{n}\mathbb{E}\left(\left(F''\left(B(\xi_i)\right)\right)\left((B_{\xi_i}-B_{u_i})^2-(\xi_i-u_i)\right)\right)^2 \leq \varepsilon
\end{aligned}$$

and the proof is done. $\square$

By substituting $H(\xi, T)$ of Theorem 5.3 to Lemma 5.1, the Itô Formula follows.

Theorem 5.6 *(Itô Formula) Let $F : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function with additional condition Δ and*

$$\lim_{t \to s} \mathbb{E}(F''(B(s)) - F''(B(t)))^4 = 0.$$

If $F'(B(t))$ is backwards Itô integrable with respect to $B(t)$. Then we have

$$\int_s^T F'(B(t))dB(t) = F(B(s)) - F(B(0)) + \frac{1}{2}\int_s^T F''(B(t))dt.$$

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