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JENSEN'S INEQUALITY AND SOME APPLICATIONS

We will first define a *convex* function.

Definition: A function $c : G \to \mathbb{R}$ where G is an open interval in $\mathbb{R}$ is called *convex* if for any two numbers x_1 and x_2 in G, the chord ℓ that connects the points $(x_1, c(x_1))$ and $(x_2, c(x_2))$ is above the graph of c. That is for every $\lambda \in [0, 1]$, with

$$x := \lambda x_1 + (1 - \lambda)x_2$$

we have

$$c(x) \leq \lambda c(x_1) + (1 - \lambda)c(x_2).$$

Alternatively, we say,

$$c[\lambda x_1 + (1 - \lambda)x_2] \leq \lambda c(x_1) + (1 - \lambda)c(x_2).$$

Some examples of convex functions on $\mathbb{R}$ are $|x|$, x^2 and e^x. Note that x^3 is not convex on $\mathbb{R}$ but it is on $(0, \infty)$. Now here are some properties of convex functions:

Lemma: If c is a convex function on an open interval (a, b) and if x_1, x_2, x_1', x_2' are points on (a, b) with

$$x_1 \leq x_1' < x_2' \quad \text{and} \quad x_1 < x_2 \leq x_2'$$

Then the chord over the interval (x_1', x_2') has a larger slope than the chord over (x_1, x_2), that is

$$\frac{c(x_2') - c(x_1')}{x_2' - x_1'} \geq \frac{c(x_2) - c(x_1)}{x_2 - x_1}$$

Lemma: If c is convex on (a, b) then c is continuous on (a, b)

Now we can state Jensen's Inequality.

Theorem: Let c be a convex function on $\mathbb{R}$ and f an integrable function on $[0, 1]$. Then

$$\int_0^1 c[f(t)]dt \geq c\left[\int_0^1 f(t)dt.\right]$$

Some Applications

1. Show that $\displaystyle\int_0^1 e^{\cos(x^2)}dx \geq e^{\left[\int_0^1 \cos(x^2)dx\right]}$.
2. Prove the AM-GM inequality, that is, $\dfrac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$ where $a_i > 0$.
3. Discrete version of Jensen's: $\lambda_1 c(a_1) + \lambda_2 c(a_2) + \ldots + \lambda_n c(a_n) \geq c[\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n]$ where $a_1, a_2, \ldots, a_n$ are in the domain of c and $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$.
4. $\left(\sum_{i=1}^k w_i t_i\right)^2 \leq \sum_{i=1}^k w_i t_i^2$ where $t_i \in (0, T)$ and $w_1 + w_2 + \ldots + w_k = 1$.