MSP CEBU **EMMANUEL A.CABRAL**

CALCULUS APPLICATIONS IN PROBABILITY AND STATISTICS

Organization of Lecture

1. Sample Spaces and Random Variables
2. Continuous Random Variables
3. Probability Density and Cumulative Distribution Functions
4. Expectation and Variance
5. Moment-Generating Functions
6. Joint pdf and joint cdf
7. Covariance and Correlation
8. Best Linear Predictor

1 Sample Spaces and Random Variables

An *experiment* is a procedure whose outcome cannot be predicted in advance. The set of all possible outcomes of the experiment is called *sample space* Ω.

Example For a single toss of a coin, $\Omega = \{H, T\}$.

Example Two successive tosses of a coin has $\Omega = \{TT, TH, HT, HH\}$.

Example Toss a coin repeatedly until the first H appears: $\Omega = \{H, TH, TTH, \ldots\}$. This sample space is *countably infinite.*

Example Toss a coin infinitely many times. The sample space consists of infinitely long strings of H's and T's. This sample space is *uncountably infinite.*

Example Measure the actual life of a light bulb. What is the sample space?

A *random variable* X is a function $X : \Omega \to \mathbb{R}$ that assigns a real number to every outcome $\omega \in \Omega$.

Example For a single toss of a coin, we can assign $X(T) = 0$ and $X(H) = 1$.

Example For two successive coin tosses, we can make the following assignments:

ω	$X(\omega)$
TT	0
TH	1
HT	2
TT	3

2 Continuous Random Variables

Example For the infinite number of coin tosses, we can define $X(\omega)$ to be a number in the closed interval $[0, 1]$. In fact, an assignment is possible such that for every $x \in [0, 1]$, there exists an $\omega \in \Omega$ with $X(\omega) = x$. Here, X is an example of a *continuous random variable*

For our purposes, we may think of a continuous random variable as a function over the sample space whose range of values is an interval in the set of real numbers.

In many experiments, the sample space itself is an uncountably infinite set of real numbers (e.g. an interval). More often, the random variable X is an *identity mapping.* In other words, it becomes convenient to consider the outcomes as the range of the random variable itself.

Example The length of the time interval between between two successive movements of the Marikina Valley Fault, theoretically is any number in the interval $(0, \infty)$. We can take this interval as our sample space. The random variable T could be taken as $T(\omega) = \omega$ for every $\omega \in (0, \infty)$.

3 Probability Density Function and Cumulative Distribution Function

Definition Let Y be a continuous random variable. The *probability density function (pdf)* of Y is the function f_Y having the property that for any numbers a and b,

$$P(a \leq Y \leq b) = \int_a^b f_Y(y)dy.$$

Definition The cumulative distribution function (cdf) of a continuous random variable is given by

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = P(Y \leq y).$$

EXERCISES

1. The random variable X is said to have a *normal distribution* with parameters μ and σ if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)}{2\sigma}\right].$$

Show that f_X is indeed a pdf by verifying that $f_X(x) \geq 0$ for every real number x and that

$$\int_{-\infty}^{\infty} f_X(x)dx = 1.$$

2. The random variable Z is said to have a *standard normal distribution* if the parameters μ and σ are 0 and 1 respectively. Use the normal Z-table to approximate the area under the pdf curve corresponding to the given values of X:
   (a) $X \leq -2.24$
   (b) $X > 1.17$
3. For the standard normal random variable Z, we write
$$N(d) = F_Z(d) = P(Z \leq d).$$
Prove that $N(-d) = 1 - N(d)$.
4. Let z_α denote the value of Z for which $P(Z \geq z_\alpha) = \alpha$. By definition, the *interquartile range*, Q, for the standard normal curve is given by
$$Q = z_{0.25} - z_{0.75}.$$
What is the value of Q?
5. A continuous random variable Y that can take any value in the closed interval $[a, b]$ is said to have the uniform distribution if $f_X(x) = \frac{1}{(b-a)}$.
   (a) What is the probability of selecting a point from a subinterval of $[a, b]$ with length ℓ?
   (b) Suppose $W = Y^2$ and $[a, b] = [0, 1]$. Find an expression for $f_W(w)$.

4 Expectation and Variance

Definition The *Expected Value* (also known as *mean or expectation*)of a continuous random variable Y with pdf f_Y is given by
$$E(Y) = \mu_Y = \int_{-\infty}^{\infty} y f_Y(y) dy.$$

Theorem Suppose X is a continuous random variable with pdf f_X and g is a function of X, then
$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

Definition The *variance* of a random variable X with pdf f_X is given by
$$Var(X) = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_x)^2 f_X(x) dx.$$

EXERCISES

1. Consider the random variable X that follows a normal distribution with parameters μ and σ. Use the above definitions to verify the following:
   (a) $E(X) = \mu$.
   (b) $Var(X) = \sigma^2$
2. Prove that $Var(X) = E(X^2) - [E(X)]^2$

5 Moment-Generating Functions

Definition Let X be a random variable. The *moment-generating function* (mgf) for X is denoted by $M_X(t)$ and is given by

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

at all values of t where the expected value exists.

Theorem Let X be a random variable and suppose that $M_X(t)$ exists for values of t around the origin. Then for any positive integer n, we have

$$E(X^n) = M_X^{(n)}(0).$$

EXERCISES

1. Prove that the moment-generating function of a normal random variable X with mean μ and variance σ^2 is given by
$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

2. Use the moment-generating function of a normal random variable to verify the following:
   (a) $E(X) = \mu$
   (b) $Var(X) = \sigma^2$

3. The *coefficient of skewness* of a random variable's pdf is is given by
$$\gamma_1 = \frac{E(X-\mu)^3}{\sigma^3}$$
where μ and σ are the mean and standard deviation, respectively, of X. Use moment-generating function to verify that $\gamma_1 = 0$ for a normally-distributed random variable X.

Theorem Suppose that X_1 and X_2 are two random variables with $M_{X_1}(t) = M_{X_2}(t)$ for every t in some interval containing 0, then $f_{X_1}(x) = f_{X_2}(x)$.

6 Joint Pdf, Marginal Pdf, and Joint Cdf

Definition The *joint-pdf* $f_{X,Y}$ of two random variables X and Y satisfies

$$P(X \leq a; Y \leq b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x,y) dx dy.$$

Theorem Suppose X and Y are two continuous random variables with joint pdf $f_{X,Y}(x,y)$. Then the marginal pdf's of X and Y,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx, \end{aligned}$$

respectively.

EXERCISE Suppose $f_{X,Y}(x,y) = 2e^{-x}e^{-y}$ for every (x,y) in the region on the plane bounded by the y-axis and the line $y = x$ and 0 elsewhere. Find the marginal pdf of Y.

Definition The joint cdf of X and Y, denoted $F_{X,Y}$, is given by

$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v).$$

Theorem Let X, Y be random variables. Then

1. $f_X(x) = \dfrac{dF_X}{dx}(x)$.
2. $f_{X,Y} = \dfrac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y)$ provided $F_{X,Y}$ has continuous second partial derivatives.

Definition Two continuous random variables X and Y are *independent* if and only there are functions $g(x)$ and $h(y)$ such that

$$f_{X,Y}(x,y) = g(x)h(y).$$

Definition Let g be a function of the random variables X and Y. Then

$$E(g(X,Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y} dx dy.$$

So we have

$$Var(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (g(x,y) - \mu_g)^2 f_{X,Y} dx dy$$

where $\mu_g = E(g(X,Y))$.

EXERCISES

1. Suppose X and Y are independent random variables, then there are functions $g(x)$ and $h(y)$ such that
$$f_{X,Y}(x,y) = g(x)h(y).$$
Prove that there exists a constant k such that
$$\begin{aligned} f_X(x) &= kg(x) \\ f_Y(y) &= \frac{1}{k}h(y). \end{aligned}$$

2. Suppose X and Y are independent random variables, show that
$$E(XY) = E(X)E(Y).$$

3. Let X and Y be continuous random variables, and let a, b be constants. Prove the following:

(a) $E(aX + bY) = aE(X) + bE(Y)$

(b) $Var(aX) = a^2 Var(X)$

(c) $Var(X + a) = Var(X)$

(d) If X and Y are independent random variables, then

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y).$$

7 Covariance and Correlation

Definition Suppose X and Y are random variables with μ_X and μ_Y finite. The covariance of X and Y is defined by

$$\sigma_{X,Y} = Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)].$$

Theorem The covariance satisfies the following properties:

1. $Cov(X, Y) = E(XY) - E(X)E(Y)$.
2. $Cov(X, Y) = Cov(Y, X)$.
3. $Cov(X, X) = Var(X)$.
4. If $X = c$, then $Cov(X, Y) = 0$.
5. $Cov(aX + bY, Z) = aCov(X, Z) + bCov(Y, Z)$.
6. $|Cov(X, Y)| \leq \sigma_X \sigma_Y$. Equality holds if and only if either on of X or Y is constant or if there are constants a and b for which

$$Y = aX + b$$

Definition Suppose X and Y are random variables with μ_X, μ_Y both finite and σ_X, σ_Y nonzero. The *correlation coefficient* of X and Y is

$$\rho_{X,Y} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

Two random variables are said to be *uncorrelated* if $\rho_{X,Y} = 0$, *perfectly positively correlated* if $\rho_{X,Y} = 1$ and *perfectly negatively correlated* if $\rho_{X,Y} = -1$.

EXERCISES

1. Suppose X and Y are random variables and a, b are constants. Prove that

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y).$$

2. Generalize the above result so that for the random variables $X_1, X_2, \ldots, X_n$, and constants $a_1, a_2, \ldots, a_n$, we have

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j Cov(X_i, X_j).$$

8 Best Linear Predictor

Definition Let X and Y be random variables. The *best linear predictor* of Y with respect to X is the linear function $\beta X + \alpha$ that minimizes $E(\epsilon^2)$ where

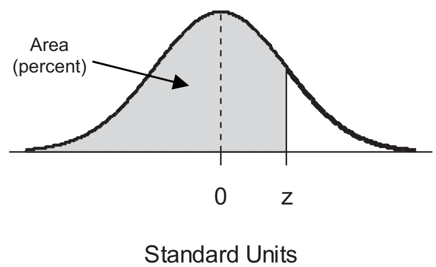
$$\epsilon = Y - \beta X - \alpha.$$

Theorem The best linear predictor of Y with respect to X is given by

$$\begin{aligned} \beta &= \frac{\sigma_{X,Y}}{\sigma_X^2} \\ \alpha &= E(Y) - \beta E(X). \end{aligned}$$

STANDARD NORMAL TABLE

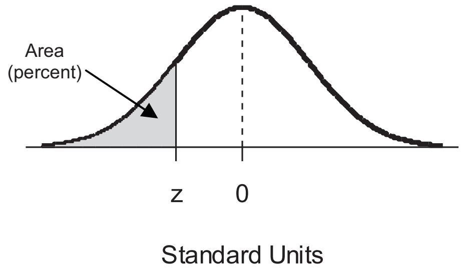
(Cumulative area to the left of POSITIVE z-values are shown in this table.)



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

STANDARD NORMAL TABLE

(Cumulative area to the left of NEGATIVE z-values are shown in this table.)



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2s981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641