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Backwards Henstock Integral

Jayrold P. Arcede
Department of Mathematics
Caraga State University
Butuan City, Philippines
jayroldparcede@yahoo.com

Emmanuel A. Cabral
Department of Mathematics
Ateneo de Manila University
Quezon City, Philippines
cabral@mathsci.math.admu.edu.ph

Abstract

In this paper, we give a proof that the backwards integral and the Lebesgue integral are equivalent.

Keywords: McShane integral, backwards Henstock integral, Henstock integral, Vitali covering

1 Introduction

In [4], an absolute integral using Vitali covers which includes the McShane non-stochastic Itô belated integral was considered. It was proved that this absolute integral and the Lebesgue integral are equivalent. In this paper, we shall discuss the *backwards Henstock integral* which is the non-stochastic version of the *backwards Itô integral* defined in $L^2(\Omega)$. Our main result is that this integral is equivalent to the Lebesgue integral. This result is crucial in proving the Itô isometry and its related results of the backward Itô integral in $L^2(\Omega)$. We begin with some preliminaries.

2 Preliminaries

Let $P = \{(u_i, v_i]\}_{i=1}^n$ be a finite collection of non-overlapping subintervals of $[a, b]$. Then P is said to be a *partial partition of* $[a, b]$. In addition, if $\bigcup_{i=1}^{n}(u_i, v_i] = (a, b]$, then P is said to be a *partition of* $[a, b]$.

Let $\delta(\xi) > 0$ on $[a, b]$, $(u, v] \subseteq [a, b]$ and $\xi \in [a, b]$, then an interval-point pair $((u, v], \xi)$ is said to be *McShane δ-fine* if $(u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$. In addition, if $\xi \in (u, v]$, then $((u, v], \xi)$ is said to be Henstock δ-fine or simply δ-fine. An interval-point pair $((u, \xi], \xi)$ is said to be backwards δ-fine if $(u, \xi] \subseteq (\xi - \delta(\xi), \xi]$.

Let $D = \{((u_i, v_i], \xi_i)\}_{i=1}^n$ be a finite collection of interval-point pairs. Then D is said to be a δ-fine partial McShane division of $[a, b]$ if $\{(u_i, v_i]\}_{i=1}^n$ is a partial division of $[a, b]$ and for each i, $((u_i, v_i], \xi_i)$, is McShane δ-fine. In addition, if $\{(u_i, v_i]\}_{i=1}^n$ is a partition of $[a, b]$, then D is said to be a δ-*fine McShane division of* $[a,b]$. A finite collection $D = \{((u_i, \xi_i], \xi_i)\}$ of interval-point pairs is said to be a *backwards* δ-*fine partial division* if $\{(u_i, \xi_i]\}_{i=1}^n$ is a partial partition of $[a,b]$ and for each i, $((u_i, \xi_i], \xi_i)$ is backwards δ-fine.

Similarly, we can define δ-fine partial (Henstock) divisions and δ-fine (Henstock) divisions of $[a,b]$. We remark that a backwards δ-fine (full) division of $[a,b]$ may not exist. For example, take $\delta(\xi) = \frac{\xi}{2}$. Then the interval $(0,T]$ is not covered by any finite collection of backwards δ-fine intervals.

Definition 2.1. *Let $f : [a,b] \to \mathbb{R}$. Then f said to be McShane integrable to A on $[a,b]$ if for every $\varepsilon > 0$, there exists a function $\delta(\xi) > 0$ on $[a,b]$ such that for every McShane δ-fine division $D = \{((u,v],\xi)\}$ of $[a,b]$ we have*

$$\left|(D)\sum f(\xi)(v-u) - A\right| < \varepsilon.$$

Note that in the above definition, we do not require $\xi \in (u,v]$.

Definition 2.2. *Let $f : [a,b] \to \mathbb{R}$. Then f is said to be Henstock integrable to A on $[a,b]$ if for every $\epsilon > 0$, there exists a function $\delta(\xi) > 0$ on $[a,b]$ such that for every δ-fine (Henstock) division $D = \{((u,v],\xi)\}$ of $[a,b]$ we have*

$$\left|(D)\sum f(\xi)(v-u) - A\right| < \varepsilon.$$

Theorem 2.1 (Henstock's lemma for Henstock integral)**.** *(See [2, p.11]) If f is Henstock integrable on $[a,b]$ with the primitive F, then for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ such that for any δ-fine division $D = ((u,v],\xi)$ we have*

$$\sum |F(v) - F(u) - f(\xi)(v-u)| < \varepsilon.$$

In the subsequent discussion, we write $F(v) - F(u)$ by $F(u,v)$.
The following is called the *Vitali Covering Theorem* (see [2]).

Theorem 2.2. *If a family of closed intervals covers a set X in the Vitali sense, then for every $\varepsilon > 0$ there is a finite number of disjoint intervals $I_1, I_2, \ldots, I_n$ in the family such that*

$$|X| \le \sum_{i=1}^{n} |I_i| + \varepsilon.$$

Recall that the outer measure of X is defined to be

$$|X| = \inf\left\{\sum_i |I_i| : \bigcup_i I_i \supset X\right\}$$

where I_i denote intervals.

Definition 2.3. *Let $[a,b]$ be a closed bounded interval of $\mathbb{R}$. Then a function $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, if for any $\varepsilon > 0$, there is a $\delta > 0$ such that for every finite collection of non-overlapping intervals $\{(a_n,b_n)\}_{i=1}^n$ in $[a,b]$ with $\sum_{i=1}^{n}(b_i - a_i) < \delta$, we have*

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon.$$

3 Backwards Henstock integral

We now define the backwards Henstock integral.

Definition 3.1. *A function f defined on $[a,b]$ is said to be backwards Henstock integrable to A if for each $\epsilon > 0$, there exist $\delta(\xi) > 0$ on $[a,b]$ and $\eta > 0$ such that for every backwards δ-fine partial division $D = \{((u,\xi],\xi)\}$ of $[a,b]$ with*

$$(D)\sum(\xi - u) > b - a - \eta$$

we have

$$\left|(D)\sum f(\xi)(\xi - u) - A\right| < \varepsilon.$$

Denote A to be $\int_a^b f$, the backwards Henstock integral of f.

Theorem 3.1. *The backwards Henstock integral is uniquely determined.*

Proof. Let f be backwards Henstock integrable to A_1 and A_2. Then for each $\varepsilon > 0$, there are positive numbers η_1 and η_2, and positive functions $\delta_1(\xi)$, $\delta_2(\xi)$ on $[a,b]$ such that

$$(D_1)\sum |f(\xi)\,(\xi - u) - A_1| < \frac{\varepsilon}{2} \tag{1}$$

and

$$(D_2)\sum |f(\xi)\,(\xi - u) - A_2| < \frac{\varepsilon}{2} \tag{2}$$

where D_1 and D_2 are δ_1 and δ_2-fine partial divisions of $[a,b]$ with

$$(D_1)\sum(\xi - u) > b - a - \eta_1$$

and

$$(D_2)\sum(\xi - u) > b - a - \eta_2,$$

respectively.

Let $\delta = \min\{\delta_1,\delta_2\}$ and $\eta = \min\{\eta_1,\eta_2\}$. Let D be a backwards δ-fine partial division of $[a,b]$ with $(D)\sum(\xi - u) > b - a - \eta$. Then D is backwards δ_1 and δ_2-fine partial division of $[a,b]$ with

$$(D)\sum(\xi - u) > b - a - \eta_1$$

and

$$(D)\sum(\xi - u) > b - a - \eta_2.$$

Hence, by inequalities (1) and (2) we get

$$\begin{aligned}|A_1 - A_2| &= |A_1 - (D)\sum f(\xi)\,(\xi - u) + (D)\sum f(\xi)\,(\xi - u) - A_2| \\ &\le |A_1 - (D)\sum f(\xi)\,(\xi - u)\,| + |(D)\sum f(\xi)\,(\xi - u) - A_2| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Therefore, $A_1 = A_2$. □

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}$ be Lebesgue integrable. Then f is backwards Henstock integrable on $[a, b]$. Moreover,*

$$\int_a^b f = (L)\int_a^b f.$$

Proof. Since f is Lebesgue integrable then f is also McShane integrable and the values of their integrals agree (see [2, p.109] and [1, p.162]). Note that Theorem 2.1 also holds true for the McShane integral with the same proof (see [2, pp.11-12]). Therefore, for each $\varepsilon > 0$, there exists a positive function $\delta(\xi)$ on $[a, b]$ such that whenever $D = \{((u, \xi], \xi)\}$ is McShane δ-fine partial division of $[a, b]$ we have

$$(D)\sum |f(\xi)(\xi - u) - F(u, \xi)| < \varepsilon$$

where $F(u, \xi)$ denotes the *McShane integral of f over $(u, \xi]$.*

Note that if $D = \{((u, \xi], \xi)\}$ is a backwards δ-fine partial division then it is a McShane δ-fine partial division. Therefore, the above statement still holds for backwards δ-fine partial division of $[a, b]$. Now F is absolutely continuous on $[a, b]$ (see [1, p.61] or [3, p.110]). Thus, by Definition 2.3, for every $\varepsilon > 0$, there exists $\eta > 0$ such that for any finite collection $D' = \{(u, \xi]\}$ of non-overlapping intervals with $(D')\sum(\xi - u) < \eta$ we have $(D')\sum |F(u, \xi)| < \varepsilon$.

Consequently, for any backwards δ-fine partial division $D = \{((u, \xi], \xi)\}$ of $[a, b]$ with $(D)\sum(\xi - u) > b - a - \eta$ we have

$$\begin{aligned} |(D)\sum f(\xi)(\xi - u) - F(a, b)| &\le (D)\sum |f(\xi)(\xi - u) - F(u, \xi)| \\ &\quad + (D_1)\sum |F((u, \xi])| \\ &\le \varepsilon + \varepsilon \\ &= 2\varepsilon, \end{aligned}$$

where $(D_1)\sum$ denotes the sum over the complement intervals $\bigcup\{(u, \xi] : ((u, \xi], \xi) \in D\}$. Hence, f is backwards Henstock integrable on $[a, b]$, and

$$\int_a^b f = (L)\int_a^b f.$$

□

Lemma 3.1 (Cauchy's Criterion)**.** *A function f is backwards Henstock integrable on $[a, b]$ if and only if for every $\varepsilon > 0$, there exist positive number η and a positive function δ on $[a, b]$ such that for any two backwards δ-fine partial divisions $D = \{((u, \xi], \xi)\}$ and $D' = \{((s, \zeta], \zeta)\}$ with*

$$(D)\sum(\xi - u) > b - a - \eta$$

and

$$(D')\sum(\zeta - s) > b - a - \eta$$

we have

$$\left|(D')\sum f(\zeta)(\zeta - s) - (D)\sum f(\xi)(\xi - u)\right| < \varepsilon.$$

Proof. ($\Rightarrow$) This direction follows easily from triangle inequality. ($\Leftarrow$) Suppose there exist a positive function $\delta_n(\xi)$ on $[a, b]$ and a positive number η_n corresponding to $\varepsilon_n = \frac{1}{n}$ for $n = 1, 2, \ldots$. Note that $\{\varepsilon_n\}$ is a decreasing sequence converging to 0. Let $\delta_{n+1} \le \delta_n$

and $\eta_{n+1} < \eta_n$ for each n. Let S_n denote $(D_n)\sum f(\xi)(\xi - u)$, where $D_n = \{((u, \xi], \xi)\}$ is backwards δ_n-fine partial division of $[a, b]$ with

$$(D_n)\sum(\xi - u) > b - a - \eta_n.$$

Here D_n, and thus S_n is fixed for each n, that is, $\{S_n\}$ is a sequence of real numbers. Now note that if $m \geq n$, then D_m is a backwards δ_n-fine partial division of $[a, b]$ and

$$(D_m)\sum(\xi - u) > b - a - \eta_m \geq b - a - \eta_n.$$

Therefore,

$$|S_n - S_m| = \left|(D_n)\sum f(\xi)(\xi - u) - (D_m)\sum f(\xi)(\xi - u)\right| < \varepsilon_n.$$

Hence, $\{S_n\}$ is a Cauchy sequence. Thus, there exists $A \in \mathbb{R}$ such that

$$\lim_{n\to\infty} S_n = A.$$

Now, given $\varepsilon > 0$, choose n with $\varepsilon_n < \varepsilon$ and $|S_n - A| < \varepsilon$. Then for each backwards δ_n-fine partial division $D = \{((u, \xi], \xi)\}$ of $[a, b]$ with

$$(D)\sum(\xi - u) > b - a - \eta_n,$$

we have

$$\begin{aligned}\left|(D)\sum f(\xi)(\xi - u) - A\right| &\leq \left|(D)\sum f(\xi)(\xi - u) - S_n\right| + |S_n - A|\\ &< \varepsilon_n + \varepsilon\\ &< \varepsilon + \varepsilon\\ &= 2\varepsilon.\end{aligned}$$

Thus, f is backwards Henstock integrable on $[a, b]$. □

Proposition 3.1. *Let f be backwards Henstock integrable on $[a, c]$ and $[c, b]$. Then f is backwards Henstock integrable on $[a, b]$. Furthermore*

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof. Let f be backwards Henstock integrable on $[a, c]$ and $[c, b]$. Then for each $\varepsilon > 0$, there exist a positive function δ_1 and a positive number η_1 such that for every backwards δ_1-fine partial division $D_1 = \{((u, \xi], \xi)\}$ of $[a, c]$ with

$$\sum(\xi - u) > b - a - \eta_1$$

we have

$$\left|(D_1)\sum f(\xi)(\xi - u) - \int_a^c f\right| \leq \frac{\varepsilon}{2}.$$

Similary, there exist a positive function δ_2 on $[c, b]$ and a positive number η_2 such that for every backwards δ_2-fine partial division $D_2 = \{((u, \xi], \xi)\}$ of $[c, b]$ with

$$\sum(\xi - u) > b - a - \eta_2$$

we have,

$$\left|(D_1)\sum f(\xi)(\xi-u)-\int_c^b f\right|\le\frac{\varepsilon}{2}.$$

Define a positive function δ on $[a,b]$ as follows:

$$\delta(\xi)=\begin{cases}\delta_1(\xi) & \text{if } \xi\in[a,c]\\ \min\{\delta_2(\xi),\xi-c\} & \text{if } \xi\in(c,b];\end{cases}$$

also, let $\eta=\eta_1+\eta_2$.

Now, $D_1=\{((u,\xi],\xi)\in D:(u,\xi]\subseteq[a,c]\}$ is a backwards δ_1-fine partial division of $[a,c]$ with $(D_1)\sum(\xi-u)>b-a-\eta_1$ and $D_2=\{((u,\xi],\xi)\in D:(u,\xi]\subseteq[c,b]\}$ is a backwards δ_2-fine partial division of $[c,b]$ with

$$\sum(\xi-u)>b-a-\eta_2.$$

Then

$$\begin{aligned}\left|(D)\sum f(\xi)(\xi-u)-\left(\int_a^c f+\int_c^b f\right)\right| &\le\left|(D_1)\sum f(\xi)(\xi-u)-\int_a^c f\right|+\\ &\qquad\left|(D_2)\sum f(\xi)(\xi-u)-\int_c^b f\right|\\ &\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &=\varepsilon.\end{aligned}$$

□

Theorem 3.3. *If f is backwards Henstock integrable on $[a,b]$, then f is backwards Henstock integrable on each closed subinterval of $[a,b]$.*

Proof. Let $[c,d]\subset[a,b]$. Suppose f is backward Henstock integrable on $[a,b]$. Then the Cauchy condition in Lemma 3.1 holds. Then given $\varepsilon>0$, there exists a positive number η and a positive function δ on $[a,b]$ such that whenever $D=\{((u,\xi],\xi)\}$ and $D'=\{((s,\zeta],\zeta)\}$ are two backwards δ-fine partial divisions of $[a,b]$ with

$$(D)\sum(\xi-u)>b-a-\eta$$

and

$$(D')\sum(\zeta-s)>b-a-\eta$$

we have

$$\left|(D')\sum f(\zeta)(\zeta-s)-(D)\sum f(\xi)(\xi-u)\right|<\epsilon. \tag{3}$$

Suppose $|c-d|>\frac{\eta}{2}$ and $|a-b|-|c-d|>\frac{\eta}{2}$. Take two backwards δ-fine partial divisions D_1,D_2 of $[c,d]$ given by $D_i=\{((u^{(i)},\xi^{(i)}],\xi^{(i)})\}$ for $i=1,2$ with

$$(D_i)\sum(\xi^{(i)}-u^{(i)})>|c-d|-\frac{\eta}{2}.$$

Take another backwards δ-fine partial division $D_3 = \{((u^{(3)}, \xi^{(3)}], \xi^{(3)})\}$ of $[a, c] \cup [d, b]$ with

$$(D_3)\sum(\xi^{(3)} - u^{(3)}) > |c - a| + |b - d| - \frac{\eta}{2}.$$

Now, let $S_i = (D_i)\sum f(\xi^{(i)})(\xi^{(i)} - u^{(i)})$ for $i = 1, 2, 3$. Then $D = D_1 \cup D_3$ and $D' = D_2 \cup D_3$ form two backwards δ-fine partial divisions of $[a, b]$ with

$$(D_1 \cup D_3)\sum(\xi - u) > b - a - \eta$$

and

$$(D_2 \cup D_3)\sum(\zeta - s) > b - a - \eta.$$

The Riemann sum of f over $D_i \cup D_3$ is $S_i + S_3$ for $i = 1, 2$. Hence, by Cauchy condition (3) above,

$$\begin{aligned} |S_1 - S_2| &= |S_1 + S_3 - (S_2 + S_3)| \\ &= |(D_1 \cup D_3)\textstyle\sum f(\xi)(\xi - u) - (D_2 \cup D_3)\sum f(\xi)(\zeta - s)| \\ &< \varepsilon \end{aligned}$$

Thus, f is backwards Henstock integrable on $[c, d]$. □

Lemma 3.2 (Henstock's lemma for backwards Henstock Integral)**.** *Let f be backwards Henstock integrable on $[a, b]$. Then for every $\varepsilon > 0$ there exists a positive function $\delta(\xi)$ on $[a, b]$ such that for any backwards δ-fine partial division $D = \{((u, \xi], \xi)\}$ of $[a, b]$ we have*

$$(D)\sum|f(\xi)(\xi - u) - F(u, \xi)| < \varepsilon,$$

where $F(u, \xi) = \int_u^\xi f$ denotes the backwards Henstock integral of f over $(u, \xi]$.

Proof. Let f be backwards Henstock integrable on $[a, b]$. By Definition 3.1, it follows that for each $\varepsilon > 0$, there exists $\eta > 0$ and $\delta(\xi) > 0$ on $[a, b]$ such that for every backwards δ-fine partial division $D_0 = \{((u, \xi], \xi)\}$ of $[a, b]$ with

$$(D_0)\sum(\xi - u) > b - a - \eta$$

we have

$$\left|(D_0)\sum f(\xi)(\xi - u) - F(a, b)\right| < \varepsilon. \tag{4}$$

Let $D = \{((u, \xi], \xi)\}$ be a backwards δ-fine partial division of $[a, b]$. Let $E_1 = \bigcup\{(u, \xi] : ((u, \xi], \xi) \in D\}$, that is, E_1 is the union of $(u, \xi]$ from D. Furthermore, let E_2 be the closure of $[a, b] \setminus E_1$. Then E_2 consists of a finite number of disjoint closed intervals $[a_i, b_i]$, $i = 1, 2, \ldots, N$.

By Theorem 3.3, f is backward Henstock integrable over each $[a_i, b_i]$. Hence, there exists η_i with $0 < \eta_i < \frac{\eta}{N}$ and $\delta_i(\xi) > 0$ on $[a_i, b_i]$ such that for each backwards δ_i-fine partial division $D_i = \{((u^{(i)}, \xi^{(i)}], \xi^{(i)})\}$ of $[a_i, b_i]$ with

$$(D_i)\sum(\xi^{(i)} - u^{(i)}) > b_i - a_i - \eta_i,$$

we have

$$\left|(D_i)\sum f(\xi^{(i)})(\xi^{(i)} - u^{(i)}) - F(a_i, b_i)\right| < \frac{\varepsilon}{N}. \tag{5}$$

For each i, let D_i be fixed. Then

$$(D)\sum(\xi-u)+\sum_i(D_i)\sum(\xi^{(i)}-u^{(i)})>b-a-\eta,$$

and $D'=\bigcup_i D_i\cup D$ is a backward δ-fine partial division of $[a,b]$.

Observe that

$$\begin{aligned}&(D')\sum f(\xi)(\xi-u)-F(a,b)\\&\quad=(D)\sum[f(\xi)(\xi-u)-F(u,\xi)]+\\&\quad\sum_i\Big[(D_i)\sum f(\xi^{(i)})(\xi^{(i)}-u^{(i)})-F(a_i,b_i)\Big].\end{aligned}$$

Therefore by equations (4) and (5)

$$\begin{aligned}\Big|(D)\sum\{f(\xi)(\xi-u)-F(u,\xi)\}\Big|&\le\Big|(D')\sum f(\xi)(\xi-u)-F(a,b)\Big|+\\&\quad\left|\sum_i\Big[(D_i)\Big(\sum f(\xi)(\xi-u)-F(a_i,b_i)\Big)\Big]\right|\\&<\varepsilon+\frac{\varepsilon}{N}(N)\\&=2\varepsilon.\end{aligned}\tag{6}$$

Finally, let $\mathcal{D}^+=\{((u,\xi],\xi)\in D: f(\xi)(\xi-u)-F(u,\xi)\ge 0\}$ and $\mathcal{D}^-=D\setminus\mathcal{D}^+$. Then both $\mathcal{D}^+$ and $\mathcal{D}^-$ are backwards δ-fine partial divisions of $[a,b]$. Therefore, $\mathcal{D}^+$ and $\mathcal{D}^-$ satisfy the above estimate (6). Hence,

$$\begin{aligned}(D)\sum\Big|f(x)(\xi-u)-F(u,\xi)\Big|&=\sum_{((u,\xi],\xi)\in\mathcal{D}^+}\Big\{f(x)(\xi-u)-F(u,\xi)\Big\}+\\&\quad\sum_{((u,\xi],\xi)\in\mathcal{D}^-}\Big\{F(u,\xi)-f(x)(\xi-u)\Big\}\\&<2\varepsilon+2\varepsilon=4\varepsilon.\end{aligned}$$

Hence, for any fixed backwards δ-fine partial division D of $[a,b]$, we have

$$(D)\sum|f(x)(\xi-u)-F(u,\xi)|<4\varepsilon.$$

□

The next lemma is useful in showing that the primitive of a backwards Henstock integrable function is absolutely continuous.

Lemma 3.3. *Let f be backwards Henstock integrable on $[a,b]$. Let $\varepsilon>0$. Then there exists a positive number η and a positive function $\delta(\xi)$ on $[a,b]$ such that for any backwards δ-fine partial division $D=\{((u,\xi],\xi)\}$ of $[a,b]$ with*

$$(D)\sum(\xi-u)<\eta$$

we have

$$\Big|(D)\sum f(\xi)(\xi-u)\Big|<\varepsilon\quad\textit{and}\quad\Big|(D)\sum F(u,\xi)\Big|<\varepsilon,$$

where $F(u,\xi)$ denotes the backwards integral of f over $(u,\xi]$.

Proof. Let f be backwards Henstock integrable on $[a,b]$. Then for every $\varepsilon > 0$, there exist an $\eta > 0$ and $\delta(\xi) > 0$ on $[a,b]$ such that for any backwards δ-fine partial division $D_0 = \{((u,\xi],\xi)\}$ of $[a,b]$ with $(D_0)\sum(\xi - u) > b - a - 2\eta$ we have

$$\left|(D_0)\sum f(\xi)(\xi - u) - F(a,b)\right| < \frac{\epsilon}{4}.$$

Let $D = \{((u,\xi],\xi)\}$ be a backwards δ-fine partial division of $[a,b]$ with

$$(D)\sum(\xi - u) < \eta.$$

Let $E = \cup\{(u,\xi] : ((u,\xi],\xi) \in D\}$ and E_1 be the closure of $[a,b] \setminus E$. Then the outer measure, $|E_1|$, of E_1 is greater than $[b-a-\eta]-\eta$. Choose a backwards δ-fine partial division $D' = \{((s,\zeta],\zeta)\}$ of $[a,b]$ with $\zeta \in E_1$ such that the union J of intervals from D' is in E_1 and $|E_1 \setminus J| < \eta$. This is possible since the collection of all δ-fine intervals is a Vitali cover, therefore the Lemma of Vitali applies. Hence,

$$|[a,b] \setminus J| = |(E \cup E_1) \setminus J| = |E \cup (E_1 \setminus J)| < \eta + \eta$$

and

$$|[a,b] \setminus (E \cup J)| = |E_1 \setminus J| < \eta.$$

Now, since

$$(D')\sum(\zeta - s) = |J| = (b-a) - |[a,b] \setminus J| > b - a - 2\eta;$$

it follows that

$$\left|(D')\sum f(\zeta)(\zeta - s) - F(a,b)\right| < \frac{\varepsilon}{4}.$$

Also, because

$$\begin{aligned}(D \cup D')\sum(\xi - u) &= |E \cup J| \\ &= (b-a) - |[a,b] \setminus (E \cup J)| \\ &> b - a - \eta \\ &> b - a - 2\eta\end{aligned}$$

it also follows that,

$$\left|(D)\sum f(\xi)(\xi - u) + (D')\sum f(\xi)(\xi - u) - F(a,b)\right| < \frac{\varepsilon}{4}.$$

Therefore,

$$\begin{aligned}&\left|(D)\sum f(\xi)(\xi - u)\right| \\ &\quad= \left|(D \cup D')\sum f(\xi)(\xi - u) - F(a,b) - [(D')\sum f(\xi)(\zeta - s) - F(a,b)]\right| \\ &\quad< \left|(D)\sum f(\xi)(\xi - u) + (D')\sum f(\xi)(\xi - u) - F(a,b)\right| + \\ &\quad\left|(D')\sum f(\zeta)(\zeta - s) - F(a,b)\right| \\ &\quad< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &\quad< \varepsilon,\end{aligned}$$

which is the first inequality.

Now, as a consequence of the above result and Henstock Lemma,

$$\begin{aligned}\left|(D)\sum F(u,\xi)\right| &\le \left|(D)\sum f(\xi)(\xi-u)\right| + \\ &\quad (D)\sum |f(\xi)(\xi-u)-F(u,\xi)| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

which is the second inequality. □

Lemma 3.4. *Let f be backwards Henstock integrable on $[a,b]$ with primitive $F(u,v)=\int_u^v f$. Then F is absolutely continuous on $[a,b]$.*

Proof. Let f be backwards Henstock integrable on $[a,b]$. Let $\varepsilon>0$. By Lemma 3.3, there exist $\eta>0$ and $\delta(\xi)>0$ on $[a,b]$ such that for any backwards δ-fine partial division $D=\{((u,\xi],\xi)\}$ of $[a,b]$ with $(D)\sum(\xi-u)<\eta$ we have

$$(D)\sum f(\xi)(\xi-u)<\varepsilon.$$

Let $\{[a_i,b_i]\}$ be a finite sequence of non-overlapping subintervals of $[a,b]$, with

$$\sum_i |b_i-a_i|<\eta.$$

By Theorem 3.3, f is backwards Henstock integrable on each $[a_i,b_i]$. Hence, there exist a positive number η_i and a positive function $\delta_i(\xi)$, $\delta_i(\xi)<\delta(\xi)$ on $[a_i,b_i]$ such that for every backwards δ_i-fine partial division $D_i=\left\{\left((u^{(i)},\xi^{(i)}],\xi^{(i)}\right)\right\}$ of $[a_i,b_i]$ with

$$\left|(D_i)\sum f(\xi^{(i)})(\xi^{(i)}-u^{(i)})-F(a_i,b_i)\right|<\frac{\varepsilon}{2^i},$$

where $F\left(a_i,b_i\right)$ denotes the backwards Henstock integral of f over $[a_i,b_i]$.
Fix D_i for each i, then

$$\sum_i (D_i)\sum(\xi^{(i)}-u^{(i)})\le\sum_i |b_i-a_i|<\eta,$$

and $D=\bigcup_i D_i$ is a backwards δ-fine partial division of $[a,b]$ with $\delta_i(\xi)\le\delta(\xi)$. By Lemma 3.3,

$$\left|\sum_i (D_i)\sum f(\xi^{(i)})(\xi^{(i)}-u^{(i)})\right|<\varepsilon.$$

Therefore,

$$\begin{aligned}\left|\sum_i F(a_i,b_i)\right| &\le \left|\sum_i\left[(D_i)\sum f(\xi^{(i)})(\xi^{(i)}-u^{(i)})-\sum_i F(a_i,b_i)\right]\right| \\ &\quad +\left|\sum_i (D_i)\sum f(\xi^{(i)})(\xi^{(i)}-u^{(i)})\right|. \\ &<\varepsilon+\varepsilon=2\varepsilon.\end{aligned}$$

Thus, F is absolutely continuous on $[a,b]$. □

We say that $f(\xi)$ is the *left-hand derivative at ξ of F*, denoted by $F'_-(\xi)$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(\xi)(\xi - u) - F(u,\xi)| < \varepsilon(\xi - u)$$

whenever $(u,\xi] \subset (\xi-\delta,\xi]$. On the other hand, $f(\xi)$ is the *right hand derivative at ξ of F*, denoted by $F'_+(\xi)$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(\xi)(v - \xi) - F(\xi,v)| < \varepsilon(v - \xi)$$

whenever $(\xi,v] \subset [\xi,\xi+\delta)$. If $F'_-(\xi) = F'_+(\xi)$, then we say that F has a *derivative at* ξ. We write $F'(\xi) = f(\xi)$.

Theorem 3.4. *Let f be backwards Henstock integrable on $[a,b]$ with primitive $F(u,v) = \int_u^v f$. Then*

$$F'(\xi) = f(\xi) \text{ a.e.}$$

Proof. Let f be backwards Henstock integrable on $[a,b]$ with primitive F. By Henstock Lemma, given any $\varepsilon > 0$, there exists $\delta(\xi) > 0$ on $[a,b]$ such that for any backwards δ-fine partial division $D = \{((u,\xi],\xi)\}$ of $[a,b]$ we have

$$(D)\sum |F(u,\xi) - f(\xi)(\xi - u)| < \varepsilon. \tag{7}$$

Let X be the set of points at which either $F'(\xi)$ does not exist or if it does, is not equal to $f(\xi)$. We shall prove that X is of Lebesgue measure zero. From the definition of X, we see that for every $\xi \in X$ there is an $\eta(\xi) > 0$ such that for every $\beta(\xi) > 0$ on $[a,b]$, there exists at least one backwards β-fine interval point $((u,\xi],\xi)$ with

$$|F(u,\xi) - f(\xi)(\xi - u)| > \eta(\xi)(\xi - u). \tag{8}$$

Fix n and let X_n denote the subset of X for which $\eta(\xi) \geq \frac{1}{n}$. Let A_n be the family of all β-fine interval point pairs $((u,\xi],\xi)$ that satisfy the inequality (8), with $\xi \in X_n \cap (u,\xi]$ and $\beta(\xi) < \delta(\xi)$.

Then A_n is a Vitali cover of X_n. Hence, by Lemma of Vitali, for each $\varepsilon > 0$ given in equation (7) there is a finite number of disjoint closed $[u_k,\xi_k]$, $([u_k,\xi_k],\xi_k) \in A_n$, $k = 1,\dots,m$ such that

$$|X_n| \leq \sum_{k=1}^{m}(\xi_k - u_k) + \varepsilon.$$

Note that $\{([u_k,\xi_k],\xi_k)\}$ is a backward δ-fine partial division since $\beta(\xi) < \delta(\xi)$. Therefore by inequality (8),

$$\begin{aligned} |X_n| &< \sum_{k=1}^{m}\left\{\frac{F(u_k,\xi_k) - f(\xi_k)(\xi_k - u_k)}{\eta(\xi_k)}\right\} + \varepsilon \\ &\leq m\varepsilon + \varepsilon. \end{aligned}$$

Since ε is arbitrary and m is fixed, $|X_n| = 0$. Hence, $|X| \leq \sum |X_n| = 0$. Therefore, $|X| = 0$ as desired. Therefore, $F'_-(\xi) = f(\xi)$ *a.e.* Now, by lemma 3.4, F is absolutely continuous on $[a,b]$. This implies that, $F'(\xi)$ exist *a.e*, that is, $F'_-(\xi) = F'_+(\xi)$ *a.e.* Therefore, $F'(\xi) = f(\xi)$ almost everywhere. ∎

Theorem 3.5. *If f is backwards Henstock integrable on $[a,b]$, then f is Lebesgue integrable there and*

$$\int_a^b f = (L)\int_a^b f.$$

Proof. Let f be backwards Henstock integrable on $[a,b]$. Then by Lemma 3.4, F is absolutely continuous on $[a,b]$. Hence, $F'(\xi)$ exists almost everywhere (See [3, p.109]) and

$$F(\xi,b) = (L)\int_\xi^b F'(t)dt. \tag{9}$$

Also by Theorem 3.4,

$$F'(t) = f(t) \text{ a.e.} \tag{10}$$

From (9) and (10) we get

$$F(\xi,b) = \int_\xi^b f(t)dt.$$

Therefore,

$$(L)\int_\xi^b f(t)dt = \int_\xi^b f(t)dt.$$

□

Finally, our main result follows from Theorem 3.2 and Theorem 3.5 above. We state it as a corollary.

Corollary 3.1 (Main Result)**.** *A function f defined on $[a,b]$ is Lebesgue integrable if and only if f is backwards Henstock integrable there and the values of the two integrals are equal.*

References

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