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PROCEEDINGS

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Jose Maria P. Balmaceda

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Foreword

This volume is devoted to selected papers from the 9th Philippines-Taiwan Symposium on Analysis (9th PTSOA) held on 24-26 October 2011 at the Montebello Villa Hotel, Cebu City, Philippines. The biennial symposium was first held in 1996 at the University of the Philippines Diliman and the venues have alternated between Taiwan and the Philippines. The symposium once again brought together mathematicians from the Philippines and Taiwan to share their recent results and current research activities. The activity also provided an opportunity for representatives of the Mathematical Society of the Philippines and the Mathematical Society of the Republic of China (Taiwan) to exchange ideas on the promotion and strengthening of linkages between mathematics departments of Philippine and Taiwan universities. For many participants, the Symposium was also an occasion to meet again and renew friendships.

The Mathematical Society of the Philippines is proud to be a partner in this worthwhile activity and is again publishing refereed papers from the symposium as a volume of its official journal, the Matemática Matematika. We hope that this publication will expose readers to the various researches undertaken by the authors and promote interest in the areas covered. The papers in the symposium were on various topics of pure and applied analysis, with a majority of papers dealing in several aspects of the theory of partial differential equations. We thank the authors for contributing to this volume and sincerely appreciate their cooperation and patience.

The main sponsors of the 9th Philippines-Taiwan Symposium on Analysis were the Department of Science and Technology (Philippines) and the National Science Council of the Republic of China, Taiwan. The Symposium is part of a three-year project implemented under the framework of the Taiwan-Philippines collaboration program overseen by the Taiwan Economic and Cultural Office (TECO) and the Manila Economic and Cultural Office (MECO). Support was also provided by the University of the Philippines Diliman. The Institute of Mathematics of the University of the Philippines Diliman took charge of the local organization. Publication of this volume was also partially supported by the DOST grant.

JOSE MARIA P. BALMACEDA
Issue Editor
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Fundamental Theorem of Calculus for the Backwards Itô Integral

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Abstract

In this paper, a definition of backwards stochastic differentiation is introduced. A necessary and sufficient set of conditions for backwards Itô integration and differentiation to be reversible processes is given. Backwards Itô integration is defined using the the generalized Riemann approach.

Keywords: Backwards Itô integral, backwards $L^2$-martingale, $AC^2$-property.

1 Introduction

In [2], a stochastic integral called the backwards Itô integral which assumes adaptedness property with respect to backwards filtration is introduced. This integral was defined using the generalized Riemann approach which is also called the Henstock approach. This approach was discovered independently by J. Kurzweil and R. Henstock in the 1950s to study the classical (non-stochastic) integral (see [5, 6, 9]).

A Henstock approach uses non-uniform meshes which vary from point to point unlike the uniform meshes depicted in the usual Riemann approach. This modification of the classical definition of Riemann leads to the integrals which are more general than the Riemann-Stieltjes integral and the Lebesgue-Stieltjes integral. See, for example, [5], [6] or [8].

The reader is reminded that it is impossible to define stochastic integrals using the Riemann approach since the integrators have paths of unbounded variation and the integrands are highly oscillatory. However, generalized Riemann approach has been used to define the Itô integral instead. See, for instance, [4], [7], [11], [12], [13], [14], and [15] among others. These studies showed that integrals defined by generalized Riemann approach encompass the classical stochastic integral.

In this note, we shall formulate a version of the fundamental theorem of calculus for the backwards Itô integral. The fundamental theorem characterizes the pair of processes $F$ and $f$ so that differentiation and integration become inverse operations in the following sense: differentiate $F$ to get $f$ then recover $F$ by integration and vice versa. Our characterization would involve the notion of backwards $L^2$-martingale and $AC^2$-property.
2 Preliminaries

We will assume familiarity with the definitions and basic properties that can be found in [2]. Throughout this note, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}_0^+ \) the set of nonnegative real numbers, \( \mathbb{N} \) the set of positive integers and \((\Omega, \mathcal{G}, \mathbb{P})\) denotes a probability space.

Let \( \{\mathcal{G}^s : 0 \leq s \leq T\} \) be a family of sub-\( \sigma \)-algebras of \( \mathcal{G} \). Then \( \{\mathcal{G}^s : 0 \leq s \leq T\} \) is called a backwards filtration if \( \mathcal{G}^t \subseteq \mathcal{G}^s \) for all \( 0 \leq s < t \leq T \). If in addition, \( \{\mathcal{G}^s : 0 \leq s \leq T\} \) satisfies the following condition: (1) \( \mathcal{G}^T \) contains all sets of \( \mathbb{P} \)-measure zero in \( \mathcal{G} \); and (2) for each \( s \in [0, T] \), \( \mathcal{G}^s = \mathcal{G}^{s-} \) where \( \mathcal{G}^{s-} = \bigcap_{\varepsilon > 0} \mathcal{G}^{s-\varepsilon} \). Then \( \{\mathcal{G}^s : 0 \leq s \leq T\} \) is called a standard backwards filtration. We often write \( \{\mathcal{G}^s\} \) instead of \( \{\mathcal{G}^s : 0 \leq s \leq T\} \).

A stochastic process \( f \) or simply process is a function \( f : \Omega \times [0, T] \to \mathbb{R} \), where \([0, T]\) is an interval in \( \mathbb{R}_0^+ \) and \( f(\cdot, s) \) is \( \mathcal{G}^s \)-measurable for each \( s \in [0, T] \). A process \( f = \{f_s : s \in [0, T]\} \) is said to be adapted to the standard backwards filtration \( \{\mathcal{G}^s\} \) if \( f_s \) is \( \mathcal{G}^s \)-measurable for each \( s \in [0, T] \). Let \( B = \{B_t : t \in \mathbb{R}_0^+\} \) be a Brownian motion (BM) with \( B_t - B_u \) having mean 0 and variance \( \xi - u \). Let \( \sigma(B_u : s \leq u \leq T) \) be the smallest \( \sigma \)-algebra generated by \( \{B_u : s \leq u \leq T\} \). This is the smallest \( \sigma \)-algebra containing the information about the structure of BM on \([s, T]\).

Throughout this note, we assume that the standard backwards filtration \( \{\mathcal{G}^s\} \) is the family of \( \sigma \)-algebras \( \sigma(B_u : s \leq u \leq T) \). This family is then called the natural backwards filtration of \( B \), (see [1, pp. 239-240]). Let \( (\Omega, \mathcal{G}, \{\mathcal{G}^s\}, \mathbb{P}) \) be a standard backwards filtering space. We write \( L^p(\Omega) \) for \( L^p(\Omega, \mathcal{G}, \mathbb{P}) \) where \( f \in L^p(\Omega) \) if \( E(f)^p < \infty \). For \( f \in L^1(\Omega) \), let \( E(f) \) denote the expectation of \( f \), that is, \( E(f) = \int_\Omega f \, d\mathbb{P} \). The conditional expectation of \( f \) given \( \mathcal{G}^s \) is the random variable \( E(f|\mathcal{G}^s) \).

3 Backwards Itô Integral

In this section, we shall present the backwards Itô integral and some related results. For proof the reader is referred to [3].

Let \( \delta \) be a positive function on \((0, T]\). A finite collection \( D = \{(u_i, \xi_i, \xi_i)\}_{i=1}^n \) of interval-point pairs is said to be a backwards partial division of \([0, T]\) if \( \{(u_i, \xi_i)\}_{i=1}^n \) is a finite collection of disjoint subintervals of \((0, T]\).

An interval-point pair \((u, \xi, \xi)\) is said to be backwards \( \delta \)-fine if \((u, \xi) \subseteq (\xi - \delta(\xi), \xi)\), whenever \((u, \xi) \subseteq (0, T]\) and \( \xi \in (0, T]\). We call \( D = \{(u_i, \xi_i, \xi_i)\}_{i=1}^n \) a backwards \( \delta \)-fine partial division of \([0, T]\) if \( D \) is a backwards partial division of \([0, T]\) and for each \( i \), the interval-point pair \((u_i, \xi_i, \xi_i)\) is backwards \( \delta \)-fine.

Let \( \delta > 0 \). One may not be able to find a full division that covers the entire interval \((0, T]\). For example, take \( \delta(\xi) = \frac{\xi}{2} \). Then the interval \((0, T]\) cannot be covered by any finite collection of backwards \( \delta \)-fine intervals.

Let \( \eta > 0 \) be given, a backwards \( \delta \)-fine partial division \( D \) is said to fail to cover \((0, T]\) by at most Lebesgue measure \( \eta \) if

\[
T - \sum_{i=1}^{n} (\xi_i - u_i) \leq \eta.
\]

The backwards Itô integral is defined as follows.

**Definition 3.1.** Let \( f = \{f_s : s \in [0, T]\} \) be a process adapted to the standard backwards filtering space \((\Omega, \mathcal{G}, \{\mathcal{G}^s\}, \mathbb{P})\). Then \( f \) is said to be backwards Itô integrable on \([0, T]\) if there
exists an $A \in L^2(\Omega)$ such that for any $\varepsilon > 0$, there exist a positive function $\delta$ on $(0, T]$ and a positive number $\eta$ such that for any backwards $\delta$-fine partial division $D = \{(u_i, \xi_i), \xi_i\}_{i=1}^n$ of $[0, T]$ that fails to cover $(0, T]$ by at most Lebesgue measure $\eta$ we have

$$\mathbb{E}\left(\|S(f, D, \delta, \eta) - A\|^2\right) \leq \varepsilon,$$  \hspace{1cm} (1)

where $S(f, D, \delta, \eta) = \sum_{i=1}^n f_{\xi_i}(B_{\xi_i} - B_{u_i})$. We denote $A$ by $\int_0^T f_t dB_t$.

Example 3.1. The Brownian motion $B$ is backwards Itô integrable on $[a, b]$ with respect to itself, and

$$\int_a^b B_t dB_t = \frac{1}{2} (B_b^2 - B_a^2) + \frac{1}{2} (b - a).$$

In the subsequent discussion, we assume that $[a, b] \subseteq [0, T]$. The result below is our version of Henstock's Lemma. The proof is standard in the classical theory of (non-stochastic) Henstock integration.

Lemma 3.1 (Henstock's Lemma). [3] Let $f$ be backwards Itô integrable on $[a, b]$ and $F(u, v) = \int_u^v f_t dB_t$ for any $(u, v) \subseteq [a, b]$. Then for every $\varepsilon > 0$, there exists a positive function $\delta$ on $[a, b]$ such that

$$\mathbb{E}\left(\sum_{i=1}^n |f_{\xi_i}(B_{\xi_i} - B_{u_i}) - F(u_i, \xi_i)|^2\right) \leq \varepsilon,$$  \hspace{1cm} (2)

whenever $D = \{(u_i, \xi_i), \xi_i\}_{i=1}^n$ is a backwards $\delta$-fine partial division of $[a, b]$.

In this paper, we always denote $F(u, v)$ by $F_{u, v}$. We shall give a complete description of this notation in the next section.

The following definition is a stochastic version of the absolute continuity property and can be found in [14, p.509].

Definition 3.2. Let $F = \{F_{s, T} : s \in [0, T]\}$ be a stochastic process. Then the process $F$ is said to have the $AC^2$-property if for each $\varepsilon > 0$, there exists $\eta > 0$ such that whenever $\{(u_i, v_i)\}_{i=1}^n$ is a finite collection of disjoint subintervals of $[0, T]$ with $\sum_{i=1}^n |v_i - u_i| \leq \eta$, we have

$$\mathbb{E}\left(\sum_{i=1}^n |F_{u_i, v_i}|^2\right) \leq \varepsilon.$$  

The result below is a stochastic version of the classical result in non-stochastic integration theory that a primitive function of an integral is absolutely continuous.

Theorem 3.1. [3] Let $f$ be backwards Itô integrable on $[0, T]$ with $\Phi_{u, T} = \int_u^T f_t dB_t$. Then $\Phi$ has the $AC^2$- property.

The following definition can be found in [1, p.240]

Definition 3.3. An adapted process $f = \{f_s : s \in [0, T]\}$ on $(\Omega, \mathcal{G}, \{\mathcal{G}^s\}, \mathbb{P})$ is called backwards $L^2$-martingale if i.) $\mathbb{E}(\|f_s\|) < \infty$, for all $s \in [0, T]$; ii.) $\mathbb{E}(f_t|\mathcal{G}^s) = f_s$ whenever $0 \leq t \leq s \leq T$; and iii.) $\sup_{t \in [0, T]} \mathbb{E}|f_t|^2 d\mathbb{P} < \infty$. 

The following theorem describes a backwards stochastic integral as backwards $L^2$-martingale. The proof uses Itô isometry.

**Theorem 3.2.** [3] Let $f$ be backwards Itô integrable on any subinterval $[0, T]$ of $[0, \infty)$ with $F_{(s,T]} = \int_s^T f_t dB_t$. Then the process $\{F_{(s,T]} : s \in [0,T]\}$ is backwards $L^2$-martingale with respect to its natural backwards filtration $\{\mathcal{G}^s\}$.

### 4 Differentiation

This section presents differentiation of stochastic processes in $L^2(\Omega)$. Here, the derivative in consideration is one-sided, that is, a left-sided derivative since the interval point pairs are backwards $\delta$-fine. Hence, differentiation here is left differentiation in nonstochastic integration theory. In this note, we call it backwards differentiation.

**Definition 4.1.** Let $F_{(\xi,T]}$ be a backwards stochastic process on $[0,T]$. Then $F_{(\xi,T]}$ is said to have a backwards derivative $f_\xi$, denoted as $DF_{(\xi,T]} = f_\xi$, which is a random variable at $\xi$ with respect to a Brownian motion $B$, if for each $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ such that

$$
E \left[ f_\xi(B_{\xi} - B_u) - F_{(u,\xi]} \right]^2 \leq \varepsilon E(B_{\xi} - B_u)^2 = \varepsilon(\xi - u)
$$

(3)

whenever $(u, \xi) \in \mathbb{R}$ is backwards $\delta$-fine. When $F$ has a backwards derivative at $\xi$ then $F$ is backwards differentiable at $\xi$.

We remark that in the definition above, $F_{(u,\xi]}$ is an additive interval function where $(u, \xi) \subseteq [0,T]$. That is, $F_{(u,\xi]} = F_{(u,T]} - F_{(\xi,T]}$. Very often, we also write $F(\xi)$ to mean $F_{(\xi,T]}$ so that $F_{(u,\xi]} = F(u) - F(\xi)$. Hence, an interval function can be thought of as a point function with $F(u) = F(u) - F(T)$ and $F(T) = 0$.

Usually, we impose some conditions on $F$. In this note, these conditions are the backwards $L^2$ martingale and $AC^2$ properties. Under these conditions, differentiation and integration are reversible processes. To put the main result in context, we start with a motivating example below.

Here, we will illustrate the necessity of the two conditions mentioned.

**Example 4.1.** Consider two processes $X_{(\xi,T]}$ and $Y_{(\xi,T]}$ where

$$X_{(\xi,T]} = \frac{1}{2}(B_{T}^2 - B_{\xi}^2) + \frac{1}{2}(T - \xi)$$

and

$$Y_{(\xi,T]} = \frac{1}{2}(B_{T}^2 - B_{\xi}^2)$$

are both backwards adapted processes on $[0,T]$. Note here, that the process $Y_{(\xi,T]}$ has no Itô correction, while the process $X_{(\xi,T]}$ has the Itô correction $\frac{1}{2}(T - \xi)$. We will show that $DX_{(\xi,T]} = B_{\xi} = DY_{(\xi,T]}$. We will also show that both processes have the $AC^2$ property though only $X_{(\xi,T]}$ has the $L^2$-martingale property.

$$
E \left[ B_{\xi}(B_{\xi} - B_u) - X_{(u,\xi]} \right]^2 = E \left[ \frac{1}{2}(B_{\xi} - B_u)^2 - \frac{1}{2}(\xi - u) \right]^2
$$

$$= \frac{1}{2}(\xi - u)^2
$$

$$< \frac{1}{2}\delta(\xi - u).
$$
By choosing $\delta = 2\varepsilon$, then we have

$$\mathbb{E} \left[ B_\xi (B_\xi - B_u) - X_{(u, \xi]} \right]^2 < \varepsilon(\xi - u),$$

which shows that $DX(\xi) = B_\xi$.

Similarly, we can show that $DY_{(\xi, T]} = B_\xi$. The steps are very similar to those shown above.

Now we will show that $X_{(\xi, T]}$ has the $AC^2$ property. We first note the following:

$$\mathbb{E} \left[ X_{(u, \xi]} \right]^2 \leq 2\mathbb{E} \left[ X_{(u, \xi]} - B_\xi (B_\xi - B_u) \right]^2 + 2\mathbb{E} \left[ B_\xi (B_\xi - B_u) \right]^2$$

$$\leq T(\xi - u) + 2(T + B_T^2)(\xi - u)$$

$$= C_1(\xi - u)$$

where $C_1 = 3T + 2B_T^2$. It is also useful to note the following:

$$E(B_\xi^2 - B_u^2) = E(B_\xi - B_T + B_T)^2 - E(B_u - B_T + B_T)^2$$

$$= -(\xi - u).$$

Therefore, for $u_i < \xi_i < u_j < \xi_j$, we have

$$\mathbb{E} \left[ X_{(u_i, \xi_i]} \cdot X_{(u_j, \xi_j]} \right] = 0.$$

Then for any partial division $D = \{(u_i, \xi_i)\}_{i=1}^n$ with

$$\sum_{i=1}^n (\xi_i - u_i) < \frac{\varepsilon}{C_1},$$

we have

$$\mathbb{E} \left[ \sum_{i=1}^n X_{(u_i, \xi_i]} \right]^2 \leq \sum_{i=1}^n \mathbb{E} \left( X_{(u_i, \xi_i]} \right)^2$$

$$< \varepsilon,$$

which shows that $X$ has the $AC^2$ property.

We also have

$$\mathbb{E} \left[ Y_{(u, \xi]} \right]^2 \leq 2\mathbb{E} \left[ Y_{(u, \xi]} - B_\xi (B_\xi - B_u) \right]^2 + 2\mathbb{E} \left[ B_\xi^2 (B_\xi - B_u)^2 \right]$$

$$< C_2(\xi - u)$$

where $C_2 = \frac{7}{2}T + 2B_T^2$, and

$$\mathbb{E} \left[ Y_{(u_i, \xi_i]} \cdot Y_{(u_j, \xi_j]} \right] < C_3(\xi_i - u_i)$$

where $C_3 = \frac{1}{4}T$. Thus

$$\mathbb{E} \left[ \sum_{i=1}^n Y_{(u_i, \xi]} \right]^2 < \varepsilon$$

if $D = \{(u_i, \xi_i)\}_{i=1}^n$ is a partial division with

$$\sum_{i=1}^n (\xi_i - u_i) < \frac{\varepsilon}{C_2 + 2C_3}.$$
and so $Y$ also has the $AC^2$ property.

We now investigate whether the given processes are $L^2$-martingales.

\[
\mathbb{E} \left[ B_T^2 - B_\xi^2 \right]^2 \leq \mathbb{E} \left[ B_T^2 - B_\xi^2 - 2B_T(B_T - B_\xi) \right]^2 + 2\mathbb{E} \left[ B_T(B_T - B_\xi) \right]^2 < C(T - \xi)
\]

where $C = 6T + 2B_T^2$. So

\[
\mathbb{E} |X(\xi)|^2 = \frac{1}{4} \mathbb{E} (B_T^2 - B_\xi^2)^2 - \frac{1}{2} (T - \xi)^2 + \frac{1}{4} (T - \xi)^2 \leq CT < \infty.
\]

So $\mathbb{E} |X(\xi)|^2 < \infty$, which implies that $\mathbb{E} |X(\xi)| < \infty$. Moreover, $\mathbb{E} |X(\xi)|^2$ is bounded above by $CT$ so

\[
\sup_{\xi \in [0,T]} \mathbb{E} |X(\xi)|^2 < \infty.
\]

Now, for $u < \xi$,

\[
\mathbb{E} \left[ X(u) \mid G^\xi \right] = \frac{1}{2} B_T^2 - \frac{1}{2} \mathbb{E} \left[ B_u^2 \mid G^\xi \right] + \frac{1}{2} (T - u)
\]

\[
= \frac{1}{2} B_T^2 - \frac{1}{2} \mathbb{E} \left[ (B_u - B_\xi + B_\xi)^2 \mid G^\xi \right] + \frac{1}{2} (T - u)
\]

\[
= \frac{1}{2} (B_T^2 - B_\xi^2) + \frac{1}{2} (T - \xi)
\]

\[
= X(\xi).
\]

We have shown that the process $X$ is an $L^2$-martingale. On the other hand,

\[
\mathbb{E} \left[ Y(u) \mid G^\xi \right] = Y(\xi) - \frac{1}{2} (\xi - u)
\]

\[
\neq Y(\xi),
\]

which implies that $Y$ is not an $L^2$-martingale.

In summary, we have shown that the process $X$ satisfies backwards $L^2$-martingale and $AC^2$ properties. On the other hand, the process $Y$ satisfies the $AC^2$-property on $[0,T]$ but fails to be a backwards $L^2$-martingale. These processes are not the same but have the same backward derivative. Moreover, the process $X$ displays reversibility of differentiation and integration. Hence, we impose such conditions on the main result.

## 5 Fundamental Theorem

We shall now prove the main result of this paper. Here we will show that an antiderivative of a process $f$ is the backwards Itô integral of $f$ under some specific conditions. Recall that $F$ is an antiderivative of $f$ if $DF = f$, a.e.

**Theorem 5.1.** Let $f : [0,T] \times \Omega \to \mathbb{R}$ be backwards Itô integrable on $[0,T]$ with $F(\xi,T) = \int_\xi^T f_s dB_s$. Then

1. $F$ is backwards $L^2$-martingale ;
2. $F$ has $AC^2$-property on $[0, T]$ and;

3. $DF_{[\xi, T]} = f_\xi$ a.e on $[0, T]$.

Proof. We note here that (1) and (2) follow immediately from Theorem 3.2 and Theorem 3.1. We are left to show that $DF_{[\xi, T]} = f_\xi$ a.e on $[0, T]$.

Let

$$S = \{ \xi \in [0, T] : DF_{[\xi, T]} \text{ does not exist or } DF_{[\xi, T]} \neq f_\xi \}.$$ 

Then there exists a function $\epsilon : S \to (0, 1)$ that satisfies the following: for every function $\delta : S \to (0, 1)$, there exists a family $\Gamma^\delta_\epsilon$ of point-interval pairs $((u, \xi], \xi)$ such that $(u, \xi]$ exists for every $\xi \in S$, $(u, \xi] \subset (\xi - \delta(\xi), \xi]$ and

$$E \left( f_{\xi}(B_{\xi} - B_u) - F_{(u, \xi]} \right)^2 > \epsilon(\xi)(\xi - u). \quad (4)$$

For a given $k \in \mathbb{N}$, let

$$S_k = \left\{ \xi \in S : \epsilon(\xi) \geq \frac{1}{k} \right\}.$$ 

Consider $\Gamma^\delta_\epsilon(S_k)$ to be the collection of point-interval pairs $((u, \xi], \xi) \in \Gamma^\delta_\epsilon$ such that $\xi \in S_k$.

It is useful to note here that $\Gamma^\delta_\epsilon(S_k)$ covers $S_k$ in the sense of Vitali. By Henstock's Lemma and (4), for every $\epsilon > 0$ there exists a $\delta_1 : [0, T] \to (0, 1)$ such that for every backwards $\delta_1$-fine partial division $D = \{((u_i, \xi_i], \xi_i)\}_{i=1}^n \subset \Gamma^\delta_\epsilon(S_k)$, we have

$$\frac{1}{k} \sum_{i=1}^n (\xi_i - u_i) < E \left( \sum_{i=1}^n \left| f_{\xi_i}(B_{\xi_i} - B_{u_i}) - F_{(u_i, \xi_i]} \right|^2 \right) \leq \frac{\epsilon}{2k}.$$ 

Therefore

$$\sum_{i=1}^n (\xi_i - u_i) < \frac{\epsilon}{2}.$$ 

By an application of the Vitali Covering Theorem, we can find a partial division $D = \{((u, \xi], \xi)\} \subset \Gamma^\delta_\epsilon(S_k)$ such that

$$\mu(S_k) < (D) \sum (\xi - u) + \frac{\epsilon}{2} < \epsilon.$$ 

Since $S$ is the countable union of all $S_k$, $S$ is also of Lebesgue measure zero.

The following result is the converse of the above theorem.

**Theorem 5.2.** Let $f : [0, T] \times \Gamma \to \mathbb{R}$ be a backwards adapted process on $[0, T]$. Suppose that

1. $F$ is backwards $L^2$-martingale;

2. $F$ has $AC^2$-property on $[0, T]$ and;

3. $DF_{[\xi, T]} = f_\xi$ a.e on $[0, T]$.

Then $f$ is backwards Itô integrable on $[0, T]$ with $F_{[\xi, T]} = \int_\xi^T f_s dB_s$. 

Proof. Consider the set of all $\xi \in [0, T]$ such $DF_{(\xi, T)}(\xi, T) = f_\xi$ and let $S$ be its complement in $[0, T]$. So for every $\varepsilon > 0$, there exists a function $\delta : [0, T] \to (0, 1)$ such that for $\xi \in [0, T] \setminus S$, we have
\[ \mathbb{E} \left( f_\xi(B_\xi - B_u) - F_{(u, \xi)} \right)^2 \leq \frac{\varepsilon}{2T}(\xi - u). \] (5)
whenever $(u, \xi) \subset (\xi - \delta(\xi), \xi)$. It follows that for a backwards $\delta$-fine partial division $D = \{(u, \xi), (\xi, \xi)\}$ with $\xi \in [0, T] \setminus S$, we have
\[ \mathbb{E} \left( (D) \sum |f_\xi(B_\xi - B_u) - F_{(u, \xi)}|^2 \right) < \frac{\varepsilon}{2} . \] (6)

By the $AC^2$ property, we can make
\[ \mathbb{E} \left[ (D) \sum F_{(u, \xi)} \right]^2 \leq \frac{\varepsilon}{3 \cdot 2^{k+1}} \]
by choosing $D = \{(u, \xi), (\xi, \xi)\}$ with $(D) \sum (\xi - u) < \eta_k$ for some $\eta_k$. By the backwards $L^2$-martingale property of $F$, we have
\[ \mathbb{E} \left( (D) \sum |F_{(u, \xi)}|^2 \right) < \frac{\varepsilon}{3 \cdot 2^{k+1}} . \]

Let $S_k = \{\xi \in S : m - 1 \leq |f_\xi| \leq m\}$. Each $S_k$ is of Lebesgue measure zero so we can find an open set $O_k$ containing $S_k$ with measure
\[ \mu(O_k) < \frac{\varepsilon}{3 \cdot 2k \cdot 2^{k+1}} . \]

We can choose $\eta_k < \frac{\varepsilon}{3 \cdot 2k \cdot 2^{k+1}}$ The gauge $\delta : [0, T] \to (0, 1)$ above can be chosen so that $(\xi - \delta(\xi), \xi) \subset O_k$ whenever $\xi \in S_k$. If $D = \{(u, \xi), (\xi, \xi)\}$ is a backwards $\delta$-fine partial division of $[0, T]$ with $\xi \in S_k$, then
\[ \mathbb{E} \left( (D) \sum |f_\xi(B_\xi - B_u) - F_{(u, \xi)}|^2 \right) < \frac{\varepsilon}{2^{k+1}} . \]

Finally, every backwards $\delta$-fine partial division $D = \{(u, \xi), (\xi, \xi)\}$ can be split into those that are tagged in $S$ and those tagged outside $S$. It follows that
\[ \mathbb{E} \left( (D) \sum |f_\xi(B_\xi - B_u) - F_{(u, \xi)}|^2 \right) < 2\varepsilon . \]

This proves that $f$ is backwards Itô integrable on $[0, T]$ with $F_{(\xi, T)} = \int_0^T f_s dB_s$. \hfill \Box

References


